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World-sheet instanton superpotentials in heterotic string theory and their moduli dependence

Gottfried Curio

Arnold-Sommerfeld-Center for Theoretical Physics, Department für Physik,
Ludwig-Maximilians-Universität München,
Theresienstr. 37, 80333 München, Germany

E-mail: gottfried.curio@physik.uni-muenchen.de

Abstract: To understand in detail the contribution of a world-sheet instanton to the superpotential in a heterotic string compactification, one has to understand the moduli dependence (bundle and complex structure moduli) of the one-loop determinants from the fluctuations, which accompany the classical exponential contribution (involving Kähler moduli) when evaluating the world-volume partition function. Here we use techniques to describe geometrically these Pfaffians for spectral bundles over rational base curves in elliptically fibered Calabi-Yau threefolds, and provide a (partially exhaustive) list of cases involving *factorising* (or vanishing) superpotential. This gives a conceptual explanation and generalisation of the few previously known cases which were obtained just experimentally by a numerical computation.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua

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1 Introduction and description of results

In heterotic $E_8 \times E_8$ string theory models of $N = 1$ supersymmetry in 4D arise by compactification on a Calabi-Yau threefold X with vector bundle V. Originally the case of V the tangent bundle was considered which led to an unbroken gauge group E_6 (times a hidden E_8). The generalisation to an SU(n) bundle V gives unbroken GUT groups like SO(10) and SU(5) (we will in the following focus on the visible sector and may assume an E_8 bundle V_2 embedded in the second E_8). To be able to handle the SU(n) bundle V most explicitly we assume that V arises by the spectral cover construction for bundles on an X which has an elliptic fibration $\pi : X \to B$. This description uses a surface $C \subset X$ given by an n-fold (ramified) cover of the base B and a line bundle L on C. Assuming C ample the continuous moduli of V come just from the deformations of C in X; these are given by the polynomial coefficients entering the defining equation of C.

For a holomorphic curve b, arising as support of a world-sheet instanton, to contribute to the superpotential one assumes that b is isolated and rational. In the following we want to bring to bear the explicit information about the bundle V provided by the spectral cover construction; we will therefore restrict us to the case of horizontal curves, i.e. curves b lying in B. If the world-sheet instanton contribution W_b supported on b is generically nonzero one wants to have the finer information how this contribution depends on the bundle (and possibly complex structure) moduli. This is described by the Pfaffian prefactor $Pfaff$ of the classical instanton contribution $e^{i\int_b J}$. $Pfaff$ is given by a generally complicated determinantal expression in the bundle moduli.

Because of holomorphic dependence crucial is here the sublocus in the moduli space where $Pfaff$ vanishes. This leads to an identification of $Pfaff$ with a geometrical determinantal expression whose vanishing controls the vanishing of $Pfaff$. In some cases $Pfaff$ vanishes identically in the moduli for well-understood reasons. More interesting is the case where $Pfaff$ is generically nonvanishing in the moduli. Especially interesting, and the raison d'être of the present paper, is the case where the Pfaffian shows some structure, i.e. is not as complicated as generically under the given circumstances. This means more precisely that one has a nontrivial factorisation like $Pfaff = f^k$ with $k > 1$, or $Pfaff = fg$ or even $Pfaff = f^k g$. This can simplify the search for zeroes of the superpotential, and in a case with a multiplicity $k > 1$ also of its derivative.

A small set of three examples of such behavior was found [8] by using computer calculation of some large determinants. This had the character (like a fourth example of vanishing Pfaffian) of a surprising simplification arising by an intransparent (purely numerical) brute-force computation. Our goal here is to get a conceptual understanding (i.e. beyond doing algebra for concrete matrix expressions) of the way such a simplifying structure arises. In the present paper we explain the case of the vanishing Pfaffian and the occurrence of the factor f; here explaining means we give a conceptual, non-computational reason for the examples and generalise them to further cases; the question of multiplicity k of f will be dealt with in a separate paper as will be the treatment of the somewhat differently behaved second factor g (the Giant Q_{11} in example 1, c.f. below).

In the rest of this introduction we will first recall more precisely the numerical results of [8]; then we will describe what is derived conceptually in the present paper and state the generalisations. In section 2 we recall some facts about the spectral cover construction of bundles. In section 3 we apply this to our case by restricting the construction to the elliptic surface $\mathcal{E}$ lying above the instanton curve $b \subset B$ in the elliptic fibration, providing a spectral curve c and a line bundle l on it. We investigate some special loci in moduli space for one of our main example classes, the SU(3) bundles. In section 4 we recall from [1] and [8] some general conditions for world-sheet instantons to contribute to the superpotential and make these conditions explicit in the parameters of the construction.

In section 5 we explain the main idea of the paper: how a factor in the Pfaffian can be explained by reduction to a different line bundle $\bar{l}$ which is 'simpler' than l. In section 6 we describe how this idea can be practically implemented if one can imposes a certain ('equality') condition such that the resulting question for $\bar{l}$ is again controlled by a determinantal expression. Here we give an (in a certain sense) exhaustive list of cases where the reasoning described applies. In section 7 we describe how the case of a generically vanishing Pfaffian fits into the set-up outlined so far. In section 8, which has the character of an insert, we show how the argument described so far has to be supplemented if some assumptions are weakened; this will be relevant for example 1. In section 9 we discuss in detail the examples of [8] and corresponding generalisations to which they give rise.

The appendices collect auxiliary investigations. Appendix A gives the polynomial factors of $Pfaff$ in the examples. Appendix B collects facts about (horizontal) rational curves in the Calabi-Yau threefold X. Appendix C gives Lemmata conc. the cohomological contribution criteria for instantons. Appendix D points to a different cohomological method, alternative to the procedure in section 5. Appendix E shows how the case of a negative bundle parameter λ is related to the usual procedure for $-\lambda$ (cf. example 1). Appendix F shows how one obtains the exhaustive lists for cases of reduction or vanishing Pfaffian. Appendix G gives details connecting the structural investigations and concrete matrix representations. Appendix H illustrates the special case $\chi = 0$ and shows a case of $Pfaff \equiv 0$ algebraically. Appendix I establishes needed facts and notation for symmetrized tensor products and resultants.

1.1 Some experimental results

Some examples were found experimentally via computer evaluations of the occurring large determinants [8], cf. table 1. The cases concerned spectral SU(3) bundles of bundle parameters $\lambda \in \frac{1}{2}\mathbf{Z}$ and $r = \eta b \in \mathbf{Z}$ (they are defined in section 2 and 3.2 and are needed here only to display the results in an overview) with b the base of the $\mathbf{P^1}$-fibration on $B = \mathbf{F_k}$

Example	λ	r	$\mathbf{F_k}$	$Pfaff$
1	$-5/2$	4	$\mathbf{F_1}$	$f^{11}g$
2	$3/2$	5	$\mathbf{F_1}$	f^4
3	$3/2$	1	$\mathbf{F_2}$	0
4	$3/2$	2	$\mathbf{F_2}$	f^4

Table 1. The factorisations of $Pfaff$ in the Examples. The detailed polynomials expressions of the factors of $Pfaff$ are given in appendix A.

1.2 Conceptual explanations and generalisations

When we make in section 3 the transition from a description of V over X via the spectral surface C and the line bundle L on it to the corresponding notions on $\mathcal{E}$ we will, besides the spectral curve $c := C|_{\mathcal{E}}$, also introduce the corresponding line bundle $L|_c$. And as we are able, under our assumptions, to describe the line bundle L on C as a restriction $L = \underline{L}|_C$ from a line bundle $\underline{L}$ on X, we will similarly be able to describe the line bundle $L|_c$ on c as a restriction $L|_c = l|_c$ from a line bundle l on $\mathcal{E}$; actually, of course, $l = \underline{L}|_{\mathcal{E}}$. For all these relations cf. section 3.3.

We note that when we wish to emphasize the dependence on a modulus t we write V_t, C_t and so on for the corresponding objects (cf. section 1.3). The abstract modulus t will actually turn out to be a modulus of the surface C, respectively the curve c, and will be given concretely by polynomial coefficients of its defining equation (this will be studied very explicitly for the case of SU(3) bundles, cf. (3.19) and (9.13), for example).

We usually take $\chi = c_1(B) \cdot b$ to be 0 or 1, with 1 the relevant case, cf. section 3.1; the class in $\mathcal{E} = \pi^{-1}(b)$ of the spectral cover curve c over b is $ns + rF$ with $s = \sigma|_{\mathcal{E}}$ the section of the elliptic surface $\mathcal{E}$ and F the class of the elliptic fibre.

We will explain and generalise the experimental results in section 1.1. For the degenerate case of example 3 a conceptual interpretation from the existence of nontrivial sections of $l(-F)|_c$, cf. section 7 and below, will be given (besides an algebraic explanation, cf. section H). Example 1 has a somewhat exceptional status, cf. section 8.1 and 9.3. Examples 2 and 4 are covered below (full details for examples 2,3,4 are given in the sections indicated below).

Vanishing Pfaffian: $\boldsymbol{Pfaff \equiv 0}$. Let us begin with the case where the Pfaffian vanishes identically. This happens in the following cases (in section 6.3.1, 7 and appendix F.1 it is described under which assumptions these cases give an exhaustive description; a corresponding remark applies below)

- SU(n) bundles with $\chi = 0, r = 1, \lambda \in \frac{1}{2} + \mathbf{Z}^{\geq 1}$
- SU(3) bundles with $r \not\equiv \chi(2)$ and $\lambda = 3/2$

This constitutes a vast generalisation of example 3.

Factorising Pfaffian: $Pfaff = fg$ (including the case $g = f^m$). In the main case the Pfaffian factorises such that $f|Pfaff$. More precisely we will identify a concrete factor $f = \det \bar{\iota}_1$ in $Pfaff = \det \iota_1$ where[1] (cf. section 6.3.2 and appendix F.2)

$$\iota_1 : H^1\Big(\mathcal{E}, l(-F-c)\Big) \longrightarrow H^1\Big(\mathcal{E}, l(-F)\Big), \quad l(-F) = \mathcal{O}_{\mathcal{E}}\left(n\left(\lambda + \frac{1}{2}\right)s + \beta F\right) \tag{1.1}$$

$$\bar{\iota}_1 : H^1\Big(\mathcal{E}, \bar{l}(-F-c)\Big) \longrightarrow H^1\Big(\mathcal{E}, \bar{l}(-F)\Big), \quad \bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\Big(ns + \bar{\beta}F\Big) \tag{1.2}$$

- SU(3) bundles

$\sharp$	χ	r	λ	$l(-F) = \mathcal{O}_{\mathcal{E}}(\cdot)$	$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}(\cdot)$	$\deg Pfaff$	$\deg f$
1	0	2	$\frac{3}{2}, \frac{5}{2}, \ldots$	$3(\lambda + \frac{1}{2})s - 2\lambda F$	$3s - 2F$	$6(\lambda^2 - \frac{1}{4})$	3
2	≥ 1	5χ	$\frac{3}{2}$	$6s - F$	$3s - F$	20χ	5χ
3	χ	$\geq 5\chi, \equiv \chi(2)$	$\frac{3}{2}$	$6s - (r - 5\chi + 1)F$	$3s - (\frac{r-5\chi}{2} + 1)F$	$6r - 10\chi$	$\frac{3r-5\chi}{2}$
4	0	4	$\frac{5}{2}$	$9s - 9F$	$3s - 3F$	72	6
5	1	5	$\frac{5}{2}$	$9s - 3F$	$3s - F$	62	5

Case 3 constitutes a vast generalisation of example 2 and 4 (why case 2 is listed separately besides case 3 will only become clear later and is of no concern here).

- SU(4) bundles

$\sharp$	χ	r	λ	$l(-F) = \mathcal{O}_{\mathcal{E}}(\cdot)$	$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}(\cdot)$	$\deg Pfaff$	$\deg f$
1	≥ 1	9χ	1	$6s - F$	$4s - F$	20χ	9χ
2	0	3	$\frac{3}{2}$	$8s - 4F$	$4s - 2F$	24	4

So case 5 of SU(3) and case 1 of SU(4) have a second factor (there certainly $Pfaff \neq f^k$).

1.3 Overview and summary

As the issue in question - the vanishing behaviour of the Pfaffian prefactor (of a world-sheet instanton superpotential) in dependence on the vector bundle moduli - necessarily uses a heavy amount of algebraic-geometric notions it may be useful to provide here also a nontechnical overview of the more detailled investigations which follow in the later sections (herein we allow ourselves to give an only approximate description of various issues whose more detailed aspects are dealt with in the main text).

According to the main contribution criterion which will be recalled in section 4.2 one has as precise vanishing criterion for the Pfaffian

$$Pfaff(t) = 0 \Longleftrightarrow \Gamma\Big(b, V_t|_b \otimes \mathcal{O}_b(-1)\Big) \;\cong\; \Gamma\Big(c_t, l(-F)|_{c_t}\Big) \;\neq\; 0 \tag{1.3}$$

[1] With $\beta = \frac{r+\chi}{2} - \lambda(r - n\chi) - 1$ and $\bar{\beta} = (r + \chi) - \frac{1}{2}\frac{n+1}{n-1}r - \frac{1}{2}(r - n\chi) - 1$.

Clearly, if the line bundle[2] $L = \mathcal{O}_c(D)$ over c whose possible sections are concerned here is a tensor product $L_1 \otimes L_2 = \mathcal{O}_c(D_1) \otimes \mathcal{O}_c(D_2)$ of line bundles which themselves *have* a nontrivial section, then also their product bundle, in question here, would have such a section. For example, for one - auxiliary - factor L_2 the existence of such a section might be assured by a general argument, like that the line bundle is associated to an effective divisor; thereby the problem of existence of a nontrivial section would have been reduced from the original problem for L to the other factor L_1. Because $D - D_1$ is, as assumed, effective one might say that the problem is reduced to a smaller line bundle

$$D - D_1 \text{ effective} \qquad \longrightarrow \qquad \text{reduction from } L \text{ to } L_1 \tag{1.4}$$

Actually there will be an even more important sense of such a 'reduction in size' when going from L to L_1 described later (the relevant vanishing condition will be expressed by a determinant of a matrix of smaller size; we start to develop this technically in section 5).

Now, one of the easiest possibilities for an accessible criterion for the existence of a section would arise if for L_1 a similar determinantal expression could be found (whose vanishing controls the existence of a nontrivial section) as for L. For this one considers for L_1 similar cohomological sequences as for L and has to see whether again, under certain conditions, one can relate the space of sections in question to a map between H^1-cohomologies of line bundles over $\mathcal{E} = \pi^{-1}(b)$ where the latter spaces have *equal dimension*; if that *equality condition* (which we study in section 6) is satisfied one gets indeed again a determinantal expression which controls the issue in question here, the existence of a nontrivial section of L_1

$$\text{equality condition} \quad \longrightarrow \quad \text{determinantal expression controlling } h^0(c, L_1) \neq 0 \tag{1.5}$$

Occasionally, if another special condition is fulfilled, it may also happen that the line bundle L_1 does always[3] have a section; then, of course, the same holds for L and the Pfaffian will vanish identically on the moduli space (the corresponding *vanishing condition* is studied in section 7)

$$\text{vanishing condition} \qquad \longrightarrow \qquad Pfaff \equiv 0 \tag{1.6}$$

In carrying out this program there occurs a difficulty which deserves to be mentioned. In the reduction step one needed to show that a certain line bundle (the bundle L_2) above is related to an *effective* divisor. Now in many cases when line bundles on spectral surfaces C in X or on spectral curves c in $\mathcal{E}$ are considered it is enough to restrict attention to objects induced by restriction from the ambient space, be it X or $\mathcal{E}$. For example, in our case where line bundles over the curve $c \subset \mathcal{E}$ are relevant, by restricting generality in the manner indicated and looking only to objects like $\mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)$, where b and F are the rational base and elliptic fibre of the elliptic surface $\mathcal{E}$, the question of effectiveness reduces simply to the question whether integral coefficients are non-negative. This way

[2]The notations for this and the following line bundles are used - with their here indicated meaning - only in this subsection.

[3]Independently of any special choice of moduli (such as making a determinantal expression vanish).

of doing things suffices for some major examples, like the examples 2 and 4 which are repeatedly studied in this paper like in [8]. However there are other examples which show quite interesting behaviour where employing this restriction is not sufficient, like in the example 1. More precisely what happens is this: although L_2 arises as a restriction from $\mathcal{E}$ to c of a line bundle $L_2' = \mathcal{O}_{\mathcal{E}}(D_2')$ on $\mathcal{E}$ it is not necessary that D_2' is effective as a divisor on $\mathcal{E}$; this is only sufficient

$$D_2' \quad \text{effective} \quad \overset{\not\Longleftarrow}{\Longrightarrow} \quad D_2 = D_2'|_c \quad \text{effective} \tag{1.7}$$

(the difference between the easily applicable, but too strong condition of effectiveness of D_2' and the precise condition of effectiveness of D_2 is studied in section 8). What one needs to study actually is whether D_2 is effective as a divisor on c which clearly is more difficult. Nevertheless cases like the example 1 mentioned need for their 'factorisation reduction' the employment of such more subtle line bundles.

After having gone in the required technical detail through the steps described above in sections 5 to 8 we will apply these methods to our main examples 2 or 4 and 1 in section 9.2 and 9.3, respectively. All other cases are listed in section 6.3.2 and appendix F.2.

2 Spectral SU(n) bundles over the elliptic Calabi-Yau threefold X

In case X admits an elliptic fibration $\pi : X \to B$ with a section σ one can describe the bundle V explicitly via the spectral cover C of B: the data of an SU(n) bundle are encoded by an n-fold ramified cover surface C of the base B, which datum comes down to a class η in $H^{1,1}(B)$, and a line bundle L over C; the latter, under the standing assumption $h^{1,0}(C) = 0$, reduces to the datum given by a class γ in $H^{1,1}(C)$ (whose non-triviality is crucial to get chiral matter [4]).

2.1 The elliptic Calabi-Yau space X over the surface B

Before describing the bundle in greater detail let us first elucidate the structure of the space X. Thre threefold is actually given as a hypersurface in an ambient four-fold $\mathcal{W}_B$ which itself is defined as a $\mathbf{P^2}$-bundle over the base B

$$\mathcal{W}_B = \mathbf{P}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}) \tag{2.1}$$

where $\mathcal{L} = K_B^{-1}$. The homogeneous coordinates in the fibre $\mathbf{P^2}$ are denoted by x, y, z and X is the divisor given by vanishing of the defining equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$ where g_2 and g_3 are sections of $\mathcal{L}^4$ and $\mathcal{L}^6$.

The base B is actually either a del Pezzo surface (including $\mathbf{P^2}$ and some blow-ups), a Hirzebruch surface (plus some blow-ups) or an Enriques surface. As we will have to consider rational curves, as support of the world-sheet instanton, in B we recall in appendix B the relevant facts in this regard and list the possible cases.

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2.2 The spectral cover surface C over B

The idea of the spectral cover description of an SU(n) bundle V is to consider first the bundle over an elliptic fibre F, and then to paste together these descriptions allowing global twisting data. V (assumed to be fibrewise semistable) decomposes fibrewise as a direct sum of line bundles of degree zero, encoded by a set of n points summing up to zero in the group law: this is the point $p_0 = (0,1,0)$, the point at infinity $x = y = \infty$ in affine coordinates with $z = 1$; it is globalized by the section σ. Letting this vary over the base B one gets a hypersurface $C \subset X$, a ramified n-fold cover of B. Denoting the cohomology class of $\sigma(B) \subset X$ by σ one finds, allowing the twist by a line bundle $\mathcal{M} = \mathcal{O}_B(M)$ over B with $c_1(\mathcal{M}) = [M] = \eta \in H^{1,1}(B)$

$$C = n\sigma + \eta \tag{2.2}$$

For the n-tuple of points $\{p_i \mid i = 1, \ldots, n\}$ (on a fibre) there exists a unique (up to a factor in $\mathbf{C}^*$) meromorphic function w of divisor $(w) = \sum_{i=1}^n (p_i - p_0)$: this means in the standard convention for divisors of meromorphic functions that w has zeroes at the p_i (i.e. f would be holomorphic for (f) effective). This w is, given in inhomogeneous form, a polynomial in x and y (which have a double and triple pole at p_0, respectively)

$$w = a_0 + a_2 x + a_3 y + \ldots a_n x^{(n-3)/2} y = 0 \tag{2.3}$$

(this is for n odd;[4] for n even the last term reads $a_n x^{\frac{n}{2}}$). Globally C is given as the locus (2.3) with w a section of $\mathcal{O}(\sigma)^n \otimes \mathcal{M}$ (here $\mathcal{M}$ being understood as pulled back to X) and $a_i \in H^0(B, \mathcal{M} \otimes \mathcal{L}^{-i})$.

η has to fulfill some conditions. As the spectral cover is an actual surface one needs

$$C \quad \text{effective} \qquad (\iff \quad \eta \geq 0) \tag{2.4}$$

(i.e. η effective). Second, to guarantee [3] that V is a *stable* vector bundle, one needs [7]

$$C \ \text{irreducible} \iff \quad \{a_0 = 0\} \ \text{irreducible} \qquad (\iff \ \eta \cdot b \geq 0) \tag{2.5}$$

$$\text{and } \{a_n = 0\} = C \cdot \sigma \ \text{ effective} \quad (\iff \ \eta - nc_1 \geq 0) \tag{2.6}$$

2.2.1 The moduli of V

The isomorphism class of V in this set-up will be determined by C and a certain line bundle L over it, specified in the next subsection; its cohomology class will have to take a specific form (to get $c_1(V) = 0$). Therefore an important subclass of cases (and the one to which we restrict ourselves throughout) is given by spectral cover surfaces with the divisor C not just effective but even ample (positive)

$$C \quad \text{ample} \qquad (\implies \quad \eta - nc_1 > 0) \tag{2.7}$$

[4]For $n = 3$ in section 3.2.2 we call $a_0 = C, a_2 = B, a_3 = A$ and write $D_m = \sum_{i=0}^m d_i u^i v^{m-i}$ for $D = C, B, A$ when restricting the consideration to the projective line $b \subset B$ with its homogeneous coordinates (u, v); the subscript m denotes the degree (identified as $r, r - 2\chi, r - 3\chi$ in section 3.2) of the homogeneous polynomial D (the reader will not confuse the a_i in (2.3) with the coefficients of A).

C will then have the property $h^{1,0}(C) = 0$ (inherited from X) and so line bundles on C are characterised by their Chern classes. That is, the (continuous) bundle moduli of (X, V) are then given just by $\mathbf{P}H^0(X, \mathcal{O}_X(C))$, i.e. the different choices of C which in turn are parametrised by the polynomial coefficients of its defining equation (in addition there is a discrete parameter λ described in the next subsection).

2.3 The line bundle $L = \underline{L}|_C$ over the spectral cover surface C

One describes the SU(n) bundle V over X by a line bundle L over C

$$V = p_*(p_C^* L \otimes \mathcal{P}) \tag{2.8}$$

with $p : X \times_B C \to X$ and $p_C : X \times_B C \to C$ the projections and $\mathcal{P}$ the global variant of (a symmetrized version of) the Poincare line bundle over $F_1 \times F_2$, i.e. the universal bundle which realizes F_2 as moduli space of degree zero line bundles over F_1. L is specified by a half-integral number λ. This occurs as $c_1(V) = \pi_*\big(c_1(L) + \frac{c_1(C) - c_1}{2}\big) = 0$ implies

$$c_1(L) \;=\; -\frac{1}{2}\Big(c_1(C) - \pi_{C*}c_1\Big) + \gamma \;=\; \frac{n\sigma + \eta + c_1}{2}|_C + \gamma \tag{2.9}$$

(we will omit usually the obvious pullbacks). Here γ denotes the only generally given class in the kernel of $\pi_{C*} : H^{1,1}(C) \to H^{1,1}(B)$, i.e. ($\underline{\gamma} \in H^{1,1}(X)$)

$$\gamma \;=\; \underline{\gamma}|_C \quad \text{with} \quad \underline{\gamma} \;=\; \lambda\Big(n\sigma - (\eta - nc_1)\Big) \tag{2.10}$$

This gives precise integrality conditions for λ: if n is odd, then one needs actually $\lambda \in \frac{1}{2} + \mathbf{Z}$; if n is even, then $\lambda \in \frac{1}{2} + \mathbf{Z}$ needs $c_1 \equiv 0 \bmod 2$ and $\lambda \in \mathbf{Z}$ needs $\eta \equiv c_1 \bmod 2$.

Assuming $h^{1,0}(C) = 0$ line bundles on C are characterised by their Chern classes. Therefore one can define line bundles $\underline{\mathcal{G}}$ and $\mathcal{G}$ on X and C, respectively, by

$$c_1(\underline{\mathcal{G}}) \;=\; \underline{\gamma} \;,\;\; c_1(\mathcal{G}) \;=\; \gamma \tag{2.11}$$

(when we want to make the λ-dependence explicit we denote these by $\underline{\mathcal{G}}_\lambda$ and $\mathcal{G}_\lambda$). They are related to corresponding divisor classes (modulo linear equivalence) $\underline{G}$ and G with

$$\underline{\mathcal{G}} \;=\; \mathcal{O}_X(\underline{G}) \;\;,\;\; \mathcal{G} \;=\; \mathcal{O}_C(G) \tag{2.12}$$

i.e. one has $\underline{G} = \lambda(n\sigma - \pi^*(M + nK_B))$ and, for example, $G|_c = \lambda(ns - (r - n\chi)F)|_c$.

Note that all these considerations of $\underline{\mathcal{G}}$ and $\mathcal{G}$ apply strictly only formally as the corresponding Chern classes will, taken alone for themselves, be only half-integral in general; only the full combination in (2.9) will be integral and define a proper line bundle. Similar remarks apply to the formal decompositions written below (K_B denotes here a line bundle and also the corresponding divisor class).

Explicitly one finds for the various incarnations of the spectral line bundle

$$\underline{L} = \Big(\mathcal{O}_X(\sigma)^n \otimes \pi^*\mathcal{M} \otimes \pi^* K_B^{-1}\Big)^{1/2} \otimes \underline{\mathcal{G}} \;=\; \mathcal{O}_X\left(\frac{C + \pi^* K_B^{-1}}{2} + \underline{G}\right) \tag{2.13}$$

$$L = K_C^{1/2} \otimes \pi_C^* K_B^{-1/2} \otimes \mathcal{G} \tag{2.14}$$

Explicitly one has for the Chern class

$$c_1(\underline{L}) = n\left(\lambda + \frac{1}{2}\right)\sigma + \left(\frac{1}{2} - \lambda\right)\eta + \left(\frac{1}{2} + n\lambda\right)c_1 \tag{2.15}$$

3 Spectral SU(n) bundles over the elliptic surface $\mathcal{E}$

3.1 The elliptic surface $\mathcal{E}$ over the instanton curve b

A horizontal curve lies in two surfaces in X: in the base B and in $\mathcal{E} = \pi^{-1}b$, the elliptic surface over b. Let us define the following expressions related to the restriction to $\mathcal{E}$

$$s \;:=\; \sigma|_{\mathcal{E}} \quad , \quad \chi \;:=\; c_1 \cdot b \in \mathbf{Z} \tag{3.1}$$

(with $c_1 := c_1(B)$). With the adjunction relation $\sigma^2 = -c_1\sigma$ and (2.5) one finds

$$s^2 \;=\; -\chi \leq 0 \tag{3.2}$$

with $\chi \geq 0$ from our assumptions (B.8), (B.12). The tangent bundle decomposes over b

$$TX|_b = \mathcal{O}_b(2) \,\oplus \mathcal{O}_b(a_h) \,\oplus \mathcal{O}_b(a_v) \tag{3.3}$$

with the latter two terms comprising the normal bundle where $a_h + a_v = -2$. If b is *isolated* then $a_h = a_v = -1$. Clearly for our horizontal curve b one has

$$a_h \;=\; (b|_B)^2\,, \qquad a_v \;=\; (b|_{\mathcal{E}})^2 := s^2 = -\chi \tag{3.4}$$

The canonical class of the elliptic surface $\mathcal{E}$ is

$$K_{\mathcal{E}} = \pi_{\mathcal{E}}^*\Big(K_b + \mathcal{O}_b(\chi)\Big) = \mathcal{O}_{\mathcal{E}}\Big((\chi-2)F\Big) \implies c_1(\mathcal{E}) = (2-\chi)F \tag{3.5}$$

Equivalently the relative dualizing sheaf is

$$\omega_{\mathcal{E}/b} \;=\; K_{\mathcal{E}} \otimes \pi_{\mathcal{E}}^* K_b^{-1} \;=\; \pi_{\mathcal{E}}^*\mathcal{O}_b(\chi) \tag{3.6}$$

Here (cf. for example [5]; note that the discriminant of X over B has class $\Delta = 12c_1$)

$$\chi = \chi(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) = \frac{1}{12}e(\mathcal{E}) = c_1 \cdot b \tag{3.7}$$

So, all one needs to know about the position of b in the base B is encoded by the number χ. Beyond that we will need the rank n of the vector bundle V and its spectral data η, λ, or rather r, λ after the restriction to $\mathcal{E}$, cf. (3.10). One upshot of the discussion above is that $\chi = 1$ is the relevant case to consider; for contrast we also consider $\chi = 0$ (where the whole interpretation changes) and so keep the parameter χ manifest throughout.

For b the base in the $\mathbf{P^1}$-fibered Hirzebruch surface $\mathbf{F_k} = B$ one has the following cases (where $\chi = 2 - k \geq 0$ and $c_1(\mathcal{E}) = kF$)

B	$\mathbf{F_0}$	$\mathbf{F_1}$	$\mathbf{F_2}$
$\mathcal{E}$	$K3$	dP_9	$b \times F$

The case $B = \mathbf{P^2}$ of $c_1 = 3l$ and $c_1 \cdot l = 3$ or $c_1 \cdot 2l = 6$ gives an $\mathcal{E}$ of Euler number 36 or 72, respectively.

3.2 The spectral cover curve c over b

The spectral surface C of our bundle V being an n-fold cover of the base one has

$$\begin{array}{ccc} C & \hookrightarrow & X \\ \pi_C \downarrow \; n:1 & & \pi \downarrow \; F \\ B & = & B \end{array} \tag{3.8}$$

The support of the world-sheet instanton we consider is a rational curve b inside the base B (in turn embedded in X by the zero section σ); specifically one may think of b as given by the base $\mathbf{P^1_b}$ inside the $\mathbf{P^1_f}$-fibered surface $B = \mathbf{F_k}$. Let $\mathcal{E} := \pi^{-1}b$ be the elliptic surface over b and $c := C|_{\mathcal{E}}$ the corresponding spectral curve of $V|_{\mathcal{E}}$

$$\begin{array}{ccc} c & \hookrightarrow & \mathcal{E} \\ \pi_c \downarrow \; n:1 & & \pi_{\mathcal{E}} \downarrow \; F \\ b & = & b \end{array} \tag{3.9}$$

(which we assume irreducible as we did for C). If the whole description is restricted from the elliptic threefold $X \subset \mathcal{W}_B$ over B (where the fourfold $\mathcal{W}_B$ is the $\mathbf{P^2}$-bundle of Weierstrass coordinates over B) to the elliptic surface $\mathcal{E}$ over b one gets again the equation (2.3) for $c \subset \mathcal{E} \subset \mathcal{W}_b$ (in the threefold given by the $\mathbf{P^2}$ bundle of Weierstrass coordinates over $b \cong \mathbf{P^1}$); what was $\mathcal{L} = K_B^{-1}$ for the situation over X, becomes here $\mathcal{L}|_b = \mathcal{O}_b(\chi)$ such that now $a_i|_b \in H^0(b, \mathcal{O}_b(r - i\chi))$ (where $r := \eta \cdot b$ such that $\mathcal{M}|_b = \mathcal{O}_b(r)$, cf. below). The section s, i.e. concretely the (group-)zero point $p_0 = (x_0, y_0, z_0) = (0, 1, 0) = (z) \cap F_t$ in each fibre over a point $t = u/v \in \mathbf{P^1} = b$, consists, when restricted to c, out of $s \cdot c = r - n\chi$ points (here (z) is the locus where $z = 0$).

Let us define the following expression related to the restriction to $\mathcal{E}$

$$r := \eta \cdot b \in \mathbf{Z} \tag{3.10}$$

So $C = n\sigma + \eta$ gives $c = ns + rF$. With $\eta = xb + yf$ on $\mathbf{F_k}$ one finds $r \geq 0$ with $r = 0 \Leftrightarrow \eta = x\, b_\infty$ where $b_\infty = b + kf$. This case can occur only over $\mathbf{F_2}$ as by (2.6)

$$r \geq n\,\chi \tag{3.11}$$

For C ample (as we will assume) one gets even (such that in particular always $r > 0$)

$$r > n\,\chi \tag{3.12}$$

The integer r has an interpretation as an instanton number[5]

$$c_2(V) = \eta\sigma + \omega \Longrightarrow c_2(V|_{\mathcal{E}}) = r \tag{3.13}$$

[5] The class γ would not occur in the specification of a spectral bundle $V_{\mathcal{E}}$ over the one-dimensional base b where $\pi_{c*}: H^{1,1}(c) \to H^{1,1}(b)$ is injective, cf. [2], in accord with the fact that by (3.13) $c_2(V|_{\mathcal{E}})$ sees only the γ-free part of $c_2(V)$ and not $\omega = -\frac{n^3-n}{24}c_1^2 - \frac{n}{8}\eta(\eta - nc_1) - \frac{1}{2}\pi_*(\gamma^2) \; = \; -\frac{n^3-n}{24}c_1^2 + (\lambda^2 - \frac{1}{4})\frac{n}{2}\eta(\eta - nc_1)$.

The canonical bundle of c is given by (cf. (3.5))

$$K_c = \Big(\mathcal{O}_\mathcal{E}(c)\otimes K_\mathcal{E}\Big)|_c \;=\; \mathcal{O}_\mathcal{E}\Big(ns+(r+\chi-2)F\Big)|_c \tag{3.14}$$

For later use, cf. the comments after (3.18) below, we remark that (which is ≥ 0 under our assumptions)

$$\deg K_c^{1/2} = g_c - 1 = n\left(r-\frac{n-1}{2}\chi-1\right) \tag{3.15}$$

The cohomologically nontrivial line bundles on $\mathcal{E}$ which come from X, i.e. are of the form $\mathcal{O}_\mathcal{E}(xs+yF)$, and which become flat on c are powers of[6]

$$\Lambda := \mathcal{O}_\mathcal{E}\Big(ns-(r-n\chi)F\Big) \tag{3.16}$$

3.2.1 Moduli of the spectral curve

The coefficients of the homogeneous polynomials a_i are (after one overall scaling) moduli $m\in\mathcal{M}_\mathcal{E}(c) = \mathbf{P}H^0(\mathcal{E},\mathcal{O}_\mathcal{E}(c))$ of external motions of c in $\mathcal{E}$, that is of those part of the moduli in $\mathcal{M}_{\text{bun}}(X,V)$ which is relevant in our consideration over $\mathcal{E}$.

Behaviour over the special sublocus $\Sigma_\Lambda = \{f_\Lambda = 0\}$ of the moduli space $\mathcal{M}_\mathcal{E}(c)$. Consider in the moduli space $\mathcal{M}_\mathcal{E}(c)(\ni m)$ the specialisation locus Σ_Λ where

$$\Lambda|_c \cong \mathcal{O}_c \tag{3.17}$$

and let us assume that this locus is characterised by a (set of) condition(s) $f_\Lambda(m)=0$, say.[7] Now, one of the principal objects of our study (cf. (4.9)), the line bundle $l(-F)|_c = \mathcal{O}_\mathcal{E}\Big(n(\lambda+\frac{1}{2})s+\beta F\Big)|_c \;\cong K_c^{1/2}\otimes\Lambda|_c^\lambda$, cf. (3.26) and (4.18), becomes along Σ_Λ

$$l(-F)|_c \overset{\text{on }\Sigma_\Lambda}{\longrightarrow} \mathcal{O}_\mathcal{E}\left(\left[\left(\lambda+\frac{1}{2}\right)(r-n\chi)+\beta\right]F\right)|_c = \mathcal{O}_\mathcal{E}\left(\left(r-\frac{n-1}{2}\chi-1\right)F\right)|_c \overset{\Sigma_\Lambda}{\cong} K_c^{1/2} \tag{3.18}$$

Note that (3.18) is a bundle of integral degree on c. We did exclude here the case n even, χ odd, $\lambda\in\mathbf{Z}$ where the alternative evaluation $\mathcal{O}_\mathcal{E}(\frac{n}{2}s+\frac{r-1}{2}F)|_c$ applies. So (with the mentioned alternative evaluation understood) we see *that $l(-F)|_c$ becomes along Σ_Λ effective*: $l(-F)\cong\mathcal{O}_\mathcal{E}(mF)|_c$ where $m = r-\frac{n-1}{2}\chi-1$ (and similarly for n even).

3.2.2 The case of SU(3) bundles

Here ($C=a_0, B=a_2, A=a_3$; $m\in H^0\Big(\mathcal{E},\mathcal{O}_\mathcal{E}(3s+[m]\chi F)\Big)$ for $m=z,x,y;[m]=0,2,3$)

$$\begin{aligned} w = C_r z + B_{r-2\chi}x + A_{r-3\chi}y \;\in\; & H^0\Big(\mathcal{E},\mathcal{O}_\mathcal{E}(3s+rF)\Big) \\ \cong\; & zH^0\Big(b,\mathcal{O}(r)\Big)\oplus xH^0\Big(b,\mathcal{O}(r-2\chi)\Big)\oplus yH^0\Big(b,\mathcal{O}(r-3\chi)\Big)\end{aligned} \tag{3.19}$$

[6]For $gcd(n, r-n\chi)=1$, which is fulfilled automatically in the cases of application (where $\chi=1, n\leq 5$).

[7]This locus will in general (cf. later example 1) have codimension higher than one, so f should be interpreted as a vector-valued 'function'; i.e. two or more conditions have to be posed at the same time.

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The defining equation for c is $w = C_r(t)z + B_{r-2\chi}(t)x + A_{r-3\chi}(t)y = 0$ where $t = (u,v) \in b$ with homogeneous coordinates u,v on b, i.e. (suitable pull-backs understood) $u,v \in H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(0s+1F)\Big)$. So for the special case $n=3$ the number $r-n\chi$ of points of $s \cdot c$ coincides with the degree of $A_{r-3\chi}$, and the points are just the $r-3\chi$ points $\{p_0, t_i\} \in \mathcal{E}$ in the fibres $c_{t_i} := F_{t_i}|_c$ over the zeroes t_i of $A_{r-n\chi} = \prod_{i=1}^{r-n\chi} A_1^{(i)}$

$$s|_c = \sum_{A_{r-3\chi}(t_i)=0} \{p_0, t_i\} \tag{3.20}$$

Generically in c-moduli (for $B(t_i) \neq 0$) these are *simple*[8] points in the fibres F_{t_i} w.r.t. the intersection of $s = p_0$ with the defining line $(w)_{t_i} = \{C(t_i)z + B(t_i)x = 0\}$ of c (unequal to the line $(z)_{t_i} = \{z = 0\} \subset \mathbf{P}^2_{t_i}$): here $C(t_i)z + B(t_i)x = 0$ determines x from z, and y from the Weierstrass equation, giving the other two points $p_i^\pm$ of $c_{t_i} = \{p_0 + p_i^+ + p_i^-, t_i\}$.

The codimension one sublocus $\boldsymbol{Res(A,B) = 0}$ in $\boldsymbol{\mathcal{M}_c}$. The generic result (3.20) changes at the specialisation locus $Res(A,B) = 0$ (where $B_3(t_1) = 0$, cf. appendix I.1): the (w) line (where $w = 0$ in the $\mathbf{P}^2_{t_1}$-fibre in $\mathcal{W}_b$) becomes just the (z) line and p_0 becomes a three-fold point of c_{t_1} (where $\sim$ means linear equivalence)

$$(z)|_c = 3s|_c = \sum_i \{3p_0, t_i\} = F_{t_1}|_c + \sum_{i\geq 2}\{3p_0, t_i\} \neq F_{t_1}|_c + \sum_{i \geq 2} F_{t_i}|_c \sim (r-n\chi)F|_c \tag{3.21}$$

The codimension $\boldsymbol{r-n\chi}$ sublocus $\boldsymbol{A|B}$ or $\boldsymbol{R_i = 0, i = 1, \dots, r-n\chi}$ in $\boldsymbol{\mathcal{M}_c}$. Here we demand that *not one but all* $r-n\chi$ roots of A are in common with B, i.e. $A|B$. Now $f_\Lambda = 0$ is (implied by) a set of $r-n\chi$ conditions $R_i = 0$ which are just the $r-n\chi$ conditions $B(t_i) = 0$, i.e. $R_i = Res(B, A_1^{(i)})$: demanding that all $R_i = 0$ the previous argument gives now in each c_{t_i} a threefold $\{p_0, t_i\}$, so $\mathcal{O}_{\mathcal{E}}(3s)|_c \cong \mathcal{O}_{\mathcal{E}}((r-3\chi)F)|_c$ or $\Lambda|_c \cong \mathcal{O}_c$ (cf. section 3.2.1), so (here no converse, cf. example 4 in section 9.2.2)

$$A|B \iff R_i = 0, \forall i \quad \Longrightarrow \Lambda|_c \cong \mathcal{O}_c \tag{3.22}$$

So one has $(R_i)_i = 0 \Longrightarrow f_\Lambda = 0$. Note further that the codim $r-n\chi$ locus $\{R_i(m) = 0, \forall i\} (\subset \Sigma_\Lambda)$ is a subset of the codim 1 locus $\{Res(B,A)(m) = 0\}$, i.e. $(R_i)_i = 0 \Longrightarrow Res(B,A) = 0$, so (a power of) Res is a combination[9] (in the ideal sense) of the R_i.

3.3 The line bundle $l|_c$ over the spectral cover curve c

As a further datum describing V beyond the surface C, which encodes V just fiberwise, one has a line bundle L over C with $V = p_*(p_C^* L \otimes \mathcal{P})$. L arises in the simplest case as a restriction $L = \underline{L}|_C$ to C of a line bundle $\underline{L}$ on X. We define also a corresponding restriction $l := \underline{L}|_{\mathcal{E}}$ to $\mathcal{E}$ (with $l|_c = L|_c$). So one has the inclusions of line bundles

$$\begin{array}{ccc} L & \hookrightarrow & \underline{L} \\ \downarrow & & \downarrow \\ C & \hookrightarrow & X \end{array} \quad \text{and} \quad \begin{array}{ccc} l|_c & \hookrightarrow & l \\ \downarrow & & \downarrow \\ c & \hookrightarrow & \mathcal{E} \end{array} \tag{3.23}$$

[8] Though a three-fold touching point of the line $(z)_t$ and the elliptic curve F_t in the $\mathbf{P}^2_t$ over $t \in b$.

[9] From $R_i = 0, \forall i$ one expects an $(r-3\chi)$-fold zero of the resultant; for a representation $Res = P_{r-n\chi}(R_1, \dots, R_{r-n\chi})$, with $P_{r-n\chi}$ a homogeneous polynomial of degree $r-n\chi$ in the R_i, cf. (9.5), (9.9).

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The crucial fact is that one has for spectral bundles

$$V|_B = \pi_{C*}L \qquad \text{such that} \qquad V|_b = \pi_{c*}\, l|_c \tag{3.24}$$

Similarly to (2.13) one has with $\underline{\gamma}|_{\mathcal{E}} = \lambda\big(ns-(r-n\chi)F\big)$ and $\mathcal{L}|_b = K_B^{-1}|_b = \mathcal{O}_b(\chi)$ that[10]

$$\begin{aligned} l = \underline{L}|_{\mathcal{E}} \;&\cong\; \mathcal{O}_{\mathcal{E}}\left(\frac{c+\pi_{\mathcal{E}}^*\mathcal{O}_b(\chi)}{2} + \underline{G}|_{\mathcal{E}}\right) \;=\; \mathcal{O}_{\mathcal{E}}\left(\frac{ns+(r+\chi)F}{2}\right)\otimes \underline{\mathcal{G}}|_{\mathcal{E}} \\ \Longrightarrow \qquad l(-F) \;&\cong\; \Big(K_{\mathcal{E}}\otimes \mathcal{O}_{\mathcal{E}}(c)\Big)^{1/2}\otimes \Lambda^{\lambda} \end{aligned} \tag{3.25}$$

$$\begin{aligned} l|_c = \mathcal{O}_c\left(\frac{ns+(r+\chi)F}{2}|_c\right)\otimes\mathcal{F} = L|_c \cong \Big(K_C^{1/2}\otimes \pi_C^* K_B^{-1/2}\Big)|_c\otimes\mathcal{F} \\ \cong K_c^{1/2}\otimes \pi_c^* K_b^{-1/2}\otimes\mathcal{F} \quad\Longrightarrow\quad l(-F)|_c \;\cong K_c^{1/2}\otimes\Lambda^{\lambda}|_c = \;K_c^{1/2}\otimes\mathcal{F} \end{aligned} \tag{3.26}$$

Here we have introduced the flat (cf. below) line bundle on c given by the restriction

$$\mathcal{F} = \mathcal{G}|_c \;=\; \mathcal{O}_c\big(G|_c\big) \tag{3.27}$$

Explicitly one has for the Chern class of the line bundle l on $\mathcal{E}$

$$c_1(l) = n\left(\lambda+\frac{1}{2}\right)s + \left(\left(\frac{1}{2}-\lambda\right)r + \left(\frac{1}{2}+n\lambda\right)\chi\right)F \tag{3.28}$$

One notes that the line bundle $\mathcal{F}$ over c is flat as

$$\Big(ns-(r-n\chi)F\Big)(ns+rF) = 0 \Longrightarrow \deg\, G|_c = 0 \tag{3.29}$$

Note that the flat bundle $\mathcal{F}$ has continuous moduli corresponding to $Jac(c)$ as $h^{1,0}(c)\neq 0$ whereas we did assume $h^{1,0}(C)=0$ so that we had only the discrete twist datum γ (we will not have to treat the ambiguities of $K_c^{1/2}$ in the present paper).

4 World-sheet instantons and superpotential contribution

4.1 General case: SU(n) bundles over an instanton curve b

To set the stage we recall first some versions of the criterion for a world-sheet instanton to contribute to the superpotential W. Let us assume that V is an SU(n) bundle, embedded in the first E_8 group. A world-sheet instanton, supported on an isolated rational curve b, contributes according to the following criterion [1] (with $V|_b(-1)$ denoting $V|_b\otimes\mathcal{O}_b(-1)$)

$$W_b \neq 0 \Longleftrightarrow h^0\Big(b, V|_b(-1)\Big) = 0 \tag{4.1}$$

Considered with respect to the structure group SO($2n$) $\supset$ SU(n) (with $n\leq 8$) one has

$$V|_b = \bigoplus_{i=1}^{n}\mathcal{O}_b(\kappa_i)\oplus\mathcal{O}_b(-\kappa_i) \qquad\qquad (\kappa_i\geq 0) \tag{4.2}$$

[10] The equality to the second line in (3.26) follows from $l^2|_c\otimes\mathcal{F}^{-2}\cong\mathcal{O}_{\mathcal{E}}\Big((ns+rF)+\chi F\Big)|_c = \mathcal{O}_{\mathcal{E}}\Big((ns+rF-kF)+2F\Big)|_c \cong K_c\otimes\pi_c^*K_b^{-1}$ using (3.14).

such that $V|_b(-1) = \bigoplus_{i=1}^n \mathcal{O}_b(\kappa_i - 1) \oplus \mathcal{O}_b(-\kappa_i - 1)$ gives

$$h^0(b, V|_b(-1)) = \sum_{\kappa_i - 1 \geq 0} \kappa_i = \sum \kappa_i \tag{4.3}$$

Therefore b contributes precisely if $V|_b$ is trivial, i.e.

$$W_b \neq 0 \Longleftrightarrow V|_b = \bigoplus_{i=1}^n \mathcal{O}_b \tag{4.4}$$

A corresponding framing would give n linearly independent global sections such that

$$W_b \neq 0 \Longrightarrow h^0(b, V|_b) = n \tag{4.5}$$

(This is, of course, also directly a consequence of (4.1), cf. appendix C, Lemma 1). A counterexample to the converse of (4.5) is given by $V|_b = \mathcal{O}_b \oplus \mathcal{O}_b(1) \oplus \mathcal{O}_b(-1)$.

Considered in SU(n) (as we will henceforth do) one has

$$V|_b = \bigoplus_{i=1}^n \mathcal{O}_b(k_i) \qquad \text{with} \quad \sum k_i = 0 \quad (k_i \in \mathbf{Z}) \tag{4.6}$$

where now

$$h^0(b, V|_b(-1)) = \sum_{k_i - 1 \geq 0} k_i = \sum_{k_i \geq 0} k_i \tag{4.7}$$

such that one gets (noting $\sum k_i = 0$) again (4.4)

$$\begin{aligned} h^0(b, V|_b(-1)) = 0 &\Longleftrightarrow k_i = 0 \quad \text{for all } i \text{ with } k_i \geq 0 \\ &\Longleftrightarrow k_i = 0 \quad \text{for all } i \end{aligned} \tag{4.8}$$

4.2 Elliptic case with spectral bundles and a base curve b

Precise criterion for $W_b \neq 0$.

$$W_b \neq 0 \Longleftrightarrow 0 = h^0\Big(b, V|_b(-1)\Big) = h^0\Big(b, \pi_{c*} l|_c \otimes \mathcal{O}_b(-1)\Big) = h^0\Big(c, l(-F)|_c\Big) \tag{4.9}$$

Note that $l(-F)|_c$ occurs in the short exact sequence of sheaves on $\mathcal{E}$

$$0 \longrightarrow l(-F-c) \xrightarrow{\iota} l(-F) \longrightarrow l(-F)|_c \longrightarrow 0 \tag{4.10}$$

(cf. for the following also [8]) which gives a long exact sequence of cohomology groups

$$0 \longrightarrow H^0\Big(\mathcal{E}, l(-F-c)\Big) \xrightarrow{\iota_0} H^0\Big(\mathcal{E}, l(-F)\Big) \longrightarrow H^0\Big(c, l(-F)|_c\Big) \longrightarrow \dots \tag{4.11}$$

First, necessary criterion for $W_b \neq 0$.

$$W_b \neq 0 \quad \Longrightarrow \quad dim\ \iota_0\ H^0\Big(\mathcal{E},\, l(-F-c)\Big) \;=\; dim\ H^0\Big(\mathcal{E},\, l(-F)\Big) \tag{4.12}$$

As ι_0 is an embedding this amounts just to the condition $h^0\Big(\mathcal{E},\, l(-F-c)\Big) = h^0\Big(\mathcal{E},\, l(-F)\Big)$. From (4.12) one gets as *sufficient criterion for non-contribution of b*

$$h^0\Big(\mathcal{E},\, l(-F-c)\Big) = 0 \quad , \quad h^0\Big(\mathcal{E},\, l(-F)\Big) > 0 \quad \Longrightarrow \quad W_b = 0 \tag{4.13}$$

In the following we make a technical assumption on the bundle parameter λ (cf. below)

$$\lambda > 1/2 \tag{4.14}$$

Note that then, as recalled after (4.24), the second cohomology groups in the long exact sequence (4.11) vanish, leaving only the three H^0- and the three H^1-terms (cf. Ex. after (C.22)). By $c_1(V|_b) = 0$ and (C.5) the two terms over c have equal dimension such that

$$h^0\Big(\mathcal{E}, l(-F)\Big) - h^0\Big(\mathcal{E}, l(-F-c)\Big) = h^1\Big(\mathcal{E}, l(-F)\Big) - h^1\Big(\mathcal{E}, l(-F-c)\Big) \tag{4.15}$$

If criterion (4.12) is fulfilled there is a further, more precise assertion.

Second, (conditional,) precise criterion for $W_b \neq 0$. Let the necessary condition for contribution (4.12) be fulfilled and $\lambda > 1/2$. Then

$$W_b \neq 0 \quad \Longleftrightarrow \quad dim\ \iota_1\ H^1\Big(\mathcal{E}, l(-F-c)\Big) \;=\; dim\ H^1\Big(\mathcal{E}, l(-F)\Big) \tag{4.16}$$

This follows by noting that in the long exact sequence (which can be written here as)

$$0 \longrightarrow H^0\Big(c, l(-F)|_c\Big) \longrightarrow H^1\Big(\mathcal{E}, l(-F-c)\Big) \stackrel{\iota_1}{\longrightarrow} H^1\Big(\mathcal{E}, l(-F)\Big) \longrightarrow H^1\Big(c, l(-F)|_c\Big) \longrightarrow 0$$

the outer (by $c_1(V|_b) = 0$, cf. (C.5)), and so the inner, two spaces have equal dimension. So (4.16) amounts to ι_1 being an isomorphism (generically in the moduli).

Concretely the map ι_1 is induced from multiplication with a moduli-dependent element

$$\tilde{\iota} \in H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)\Big) \tag{4.17}$$

4.3 Evaluation of the extrinsic contribution criteria

To evaluate concretely the contribution criteria note first (by $c = ns + rF$ and (3.25))

$$l(-F) = \mathcal{O}_{\mathcal{E}}\Big(\alpha s + \beta F\Big) \tag{4.18}$$

$$l(-F-c) = \mathcal{O}_{\mathcal{E}}\Big((\alpha - n)s + (\beta - r)F\Big) \tag{4.19}$$

We use the numerical parameters α, β given by[11] (where $n\chi - r \le 0$ by (3.11))

$$\alpha = n\left(\lambda + \frac{1}{2}\right) \tag{4.20}$$

$$\beta = \beta_{n,r}^{(\chi)}(\lambda) := \frac{r+\chi}{2} - \lambda(r - n\chi) - 1 = \left(\frac{1}{2} - \lambda\right) r + \left(\frac{1}{2} + n\lambda\right)\chi - 1 \tag{4.21}$$

It will be useful to keep on record the following rewriting relating α and β

$$-(\beta+1) = \frac{\alpha - n}{n}(r - n\chi) - \frac{n+1}{2}\chi \tag{4.22}$$

Now $h^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(ps+qF)\Big)$ vanishes, if $p > 0$, just for negative q (cf. appendix C, Lemma 2)

$$p > 0: \qquad h^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(ps+qF)\Big) = h^0\Big(b, \pi_{\mathcal{E}*}\mathcal{O}_{\mathcal{E}}(ps+qF)\Big) = 0 \iff q < 0 \tag{4.23}$$

In view of (4.19) and (4.23) we will in the following assume

$$\alpha - n > 0 \qquad , \text{ i.e.} \qquad \lambda > 1/2 \tag{4.24}$$

(for $\lambda < -1/2$ cf. appendix (E)). Then sheaf cohomology groups on $\mathcal{E}$ reduce, by (C.10), (C.11), to the corresponding push-forwards (direct images) on b, cf. remark after (4.14).

Following criterion (4.13) a sufficient condition for $W_b = 0$ is by (4.23)

$$\beta \ge 0 \quad , \quad \beta - r < 0 \quad \Longrightarrow \quad W_b = 0 \tag{4.25}$$

whereas an equivalent formulation of the condition (4.12) (necessary for $W_b \neq 0$) is $\beta < 0$

$$h^0\Big(\mathcal{E}, l(-F-c)\Big) = h^0\Big(\mathcal{E}, l(-F)\Big) \quad \Longleftrightarrow \quad \beta < 0 \tag{4.26}$$

$\beta < 0$ is sufficient as then both h^0 vanish by (4.23). Remarkably, the converse holds: the h^0 in (4.26) can be equal *only* if both are zero (cf. appendix C, Lemma 3). So one gets

Third, necessary criterion for $W_b \neq 0$ (case $\lambda > 1/2$).

$$W_b \neq 0 \quad \Longrightarrow \quad \beta < 0 \quad \Longleftrightarrow \quad H^0\Big(\mathcal{E}, l(-F)\Big) = 0 \tag{4.27}$$

Remarks.

1) Note that $\beta < 0$ is automatically fulfilled for $\chi = 0$.

2) (4.27) can be formulated as a stronger bound on $\eta|_b$ ((2.5) gave already $r \ge n\chi$)

$$W_b \neq 0 \Longrightarrow \left(\lambda - \frac{1}{2}\right) r \ge \left(\frac{1}{2} + n\lambda\right)\chi \tag{4.28}$$

[11] The numerical specification of β by (4.21) (which in the end goes back just to (2.9), restricted to $\mathcal{E} \subset X$) expresses just that we tuned parameters to get $c_1(V) = 0$ (resp. the corresponding version restricted to $\mathcal{E}$). This was crucial to have $h^1(\mathcal{E}, l(-F-c)) = h^1(\mathcal{E}, l(-F))$, cf. the proof after (4.16) (working within the assumption $\alpha > n$, so that the H^2-terms vanished; for a converse cf. section 6.1.1).

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This is indeed sharper: $\frac{1/2+n\lambda}{\lambda-1/2} = \frac{(1+n)/2}{\lambda-1/2} + n > n$. So one gets as necessary condition

$$r - n\chi \geq \frac{(1+n)/2}{\lambda - 1/2}\chi \tag{4.29}$$

Let us now come back to the process of making the criteria (4.12) and (4.16) more explicit. If one has $\beta < 0$, as we will assume, then (4.12) is fulfilled by (4.26) and one has, by (4.16), to consider the map (which can be further explicated using (4.18)–(4.21))

$$H^1\Big(\mathcal{E}, l(-F-c)\Big) \xrightarrow{\iota_1} H^1\Big(\mathcal{E}, l(-F)\Big) \tag{4.30}$$

This map is induced from multiplication with an element $\tilde{\iota} \in H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(ns+rF)\Big)$ and so depends $m \in \mathcal{M}_{\mathcal{E}}(c)$ (so actually $c = c_m$). What (4.16) says is that

$$Pfaff\,(m) \;=\; 0 \Longleftrightarrow \det\,\iota_1\,(m) \;=\; 0 \tag{4.31}$$

(such that $Pfaff$ equals, up to a constant factor, $(\det \iota_1)^m$; actually $m = 1$ [8]). The moduli space $\mathcal{M}_{\mathcal{E}}(c)$ has dimension $h^0\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)\big) - 1$ which is by the index theorem

$$\dim \mathcal{M}_{\mathcal{E}}(c) \;=\; h^0\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)\big) - 1 = n(r+1) - \left(\frac{n(n+1)}{2} - 1\right)\chi - 1 \tag{4.32}$$

$$= n\left(r - \frac{n}{2}\chi\right) - \left(\frac{n}{2} - 1\right)\chi + n - 1 \tag{4.33}$$

in case c is positive (for b isolated we had $\chi = 1$). Finally the degree of the determinant in (4.31) is given (using (C.14)) by (where explicitly $\frac{\alpha(\alpha-n)}{n} = (\lambda^2 - \frac{1}{4})n$)

$$h^1\Big(\mathcal{E}, l(-F)\Big) \;=\; \alpha\Big(-(\beta+1)\Big) + \frac{\alpha(\alpha+1)}{2}\chi - \chi \;=\; \frac{\alpha(\alpha-n)}{n}\left(r - \frac{n}{2}\chi\right) - \chi \tag{4.34}$$

Another way to express the precise criterion (4.9) is (with $\sim$ linear equivalence)

$$Pfaff(m) = 0 \Longleftrightarrow h^0\Big(c, l(-F)|_c\Big) > 0 \;\Longleftrightarrow\; \alpha s + \beta F|_c \sim \text{effective} \tag{4.35}$$

So one has not only $R_i = 0, \forall i \Longrightarrow f_\Lambda = 0$ by (3.22) but also $f_\Lambda = 0 \Longrightarrow Pfaff = 0$ by the remarks after (3.18); if actually codim $\Sigma_\Lambda = 1$ (cf. the remark in footnote 7) one gets[12]

$$f_\Lambda \,|\, Pfaff \tag{4.36}$$

5 The idea of reduction

The question of contribution of the world-sheet instanton supported on b to the superpotential amounts to decide whether $h^0(c, l(-F)|_c) = 0$ or not in dependence on the moduli of c (concretely, in the case of SU(3), the coefficients of the polynomials C, B, A in the equation of c). The idea of reduction is to translate the question about $h^0(c, l(-F)|_c)$ to a simpler case. For this one introduces another ('smaller') line bundle $\bar{l}$

[12]Generally $f = 0 \Rightarrow g = 0$ gives immediately only $f^{\text{red}}|g$ where $f = \prod f_i^{k_i}$ and $f^{\text{red}} = \prod f_i$, but in our actual cases (cf. section 9 where this remark applies in various places) f will be irreducible

- leading to a map $\bar{\iota}_1$ between spaces, now of lower dimension (arguing as for $l(-F)$); now one has two interesting cases to consider
  - under a certain *equality condition* these spaces have equal dimension and one is led again to the consideration of a (moduli dependent) determinantal function $f = \det \bar{\iota}_1$
  - under a *vanishing condition* one has (universally in the moduli) $\ker \bar{\iota}_1 \neq 0$, i.e. $f \equiv 0$
- under a *reduction condition* $\bar{l}(-F)$ is related to the original $l(-F)$ such that if one has $\ker \bar{\iota}_1 \neq 0$ one gets $\ker \iota_1 \neq 0$ (both either for certain moduli or universally), i.e. the vanishing of $Pfaff$ (at a special locus or universally); that is one gets
  - $f|Pfaff$ in case the equality condition applies (as $f = \det \bar{\iota}_1$ has lower degree this gives a 'reduction' in the problem of finding zeroes of $Pfaff$)
  - $Pfaff \equiv 0$ in case the vanishing condition applies

(properly taking into account footnote 12). In the present section we will treat the reduction condition, in section 6 the equality condition and in section 7 the vanishing condition.

5.1 The reduction condition: precise version (on c)

Concerning $l(-F) = \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)$ we will assume in the following $\alpha > 0$ (actually even $\alpha > n$ by (4.24)) and the necessary condition (for contribution to the superpotential) $\beta < 0$, cf. (4.27). We consider a second line bundle $\bar{l}(-F) := \mathcal{O}_{\mathcal{E}}(\bar{\alpha} s + \bar{\beta} F)$ and denote the restrictions to c by $l(-F)|_c = \mathcal{O}_c(D)$ and $\bar{l}(-F)|_c = \mathcal{O}_c(\bar{D})$. Now assume

$$\left(\bar{l}(-F)\right)^p|_c \hookrightarrow l(-F)|_c \qquad \text{(precise reduction condition)} \tag{5.1}$$

for a positive integer p. So the relevant condition is

$$p\bar{D} \leq D, \qquad \text{i.e. } \tilde{D} = D - p\bar{D} \text{ is effective} \tag{5.2}$$

such that $t \in H^0(c, \mathcal{O}_c(D - p\bar{D})), t \neq 0$ exists. This gives then

$$\begin{aligned} s \in H^0\Big(c, \bar{l}(-F)|_c\Big) &\Longrightarrow s^p \in H^0\Big(c, \mathcal{O}_{\mathcal{E}}(p\bar{\alpha} s + p\bar{\beta} F)|_c\Big) \\ &\Longrightarrow s^p t \in H^0\Big(c, \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)|_c\Big) \end{aligned} \tag{5.3}$$

as implication for the existence of nontrivial sections. Therefore one has

$$h^0\Big(c, \bar{l}(-F)|_c\Big) > 0 \Longrightarrow h^0\Big(c, l(-F)|_c\Big) > 0 \tag{5.4}$$

Here the section t has just an auxiliary status: as the difference of D and $p\bar{D}$ is effective (this is just the assumption of reduction) a nontrivial section like t will exist in any case and the problem of existence of a nontrivial section of $l(-F)|_c$ is just reduced to the corresponding problem for $\bar{l}(-F)$ (note that neither D nor $\bar{D}$ will be effective).

Remark. One can rephrase the procedure as follows. If one has $i)$ the condition for a nontrivial section (on the lhs of (5.4)) fulfilled and $ii)$ that the reduction condition holds

$$i)\quad h^0(c,\bar{l}(-F)|_c)>0 \iff \bar{D}\sim\bar{D}'\geq 0 \iff \bar{D}+(f)\geq 0 \tag{5.5}$$

$$ii)\qquad reduction \iff D\geq p\bar{D} \tag{5.6}$$

(here $\sim$ is linear equivalence) then one gets a nontrivial section for the original bundle

$$h^0(c,l(-F)|_c)>0 \iff D\sim D'\geq 0 \iff D+(f^p)\geq p(\bar{D}+(f))\geq 0 \tag{5.7}$$

Precise reduction amounts according to (5.2) to the effectivity of $\tilde{D}$; this implies that the complete linear system $|\tilde{D}|$ (of effective divisors, linearly equivalent to $\tilde{D}$) is nonempty, what just signals the existence of a nonzero section t of $\tilde{\mathcal{F}}$ where $\tilde{\mathcal{F}}=\mathcal{O}_c(\tilde{D})$ is the line bundle on c such that $\bar{l}(-F)^p|_c\otimes\tilde{\mathcal{F}}\cong l(-F)|_c$. So reduction implies $h^0(c,\tilde{\mathcal{F}})>0$.

5.2 Strong version of the reduction condition (on $\mathcal{E}$)

Actually we will usually assume the following sharper condition on $\mathcal{E}$

$$\left(\bar{l}(-F)\right)^p \hookrightarrow l(-F) \qquad \text{(strong reduction condition)} \tag{5.8}$$

This condition will imply (5.1) and is easy to check; however is is unnecessarily sharp, i.e. it is only a sufficient condition. (5.8) amounts to the effectiveness of $(\alpha-p\bar{\alpha})s$ and $(\beta-p\bar{\beta})F$, in other words

$$p\bar{\alpha}\leq\alpha\ ,\quad p\bar{\beta}\leq\beta \tag{5.9}$$

As $\beta<0$ one therefore needs to have $\bar{\beta}<0$; we will also assume $\bar{\alpha}>0$.

6 The equality condition

The equality condition is the condition which will connect the existence of a nontrivial section of $\bar{l}(-F)|_c$ with the vanishing of a corresponding determinantal expression, in precise analogy to the corresponding phenomenon for $l(-F)|_c$, cf. (4.31).

If one finds a line bundle $\bar{l}(-F)$ fulfilling (5.1) and wants to use this to find a factor of $Pfaff$ one still has to make sure some things. The first is the possibility to control $h^0(c,\bar{l}(-F)|_c)>0$ in (5.4) again by a determinantal function like in (4.31). For this one needs an equality condition (6.2) and when this holds one will have $\det\bar{\iota}_1\mid\det\iota_1$ (modulo the remark before (4.36)) which reduces the problem of finding a zero of the polynomial $\det\iota_1$ of degree $h^1(\mathcal{E},l(-F))$ to the polynomial $\det\bar{\iota}_1$ of degree $h^1(\mathcal{E},\bar{l}(-F))$ (the degree is the sum of all degrees in the individual moduli, cf. for example (A.4)).

In this connection we note that one has to check that the lhs of (6.2) should be indeed nonvanishing. In this question one has *with the assumptions* $\bar{\alpha}>0,\bar{\beta}<0$ and (C.14) that $h^1(\mathcal{E},\bar{l}(-F))>0$ if not either i) $\bar{l}(-F)=\mathcal{O}_{\mathcal{E}}(s-F)$ or ii) $\bar{l}(-F)=\mathcal{O}_{\mathcal{E}}(\bar{\alpha}s-F)$ with $\chi=0$; i) is excluded by $\bar{\alpha}\geq n$ (cf. remark after (6.2)), ii) by (6.5) and (6.8).

6.1 Numerical evaluation of the equality condition

For $\bar{l}$ most of the arguments in subsect. 4.2 are not applicable: one has neither $h^0(c, \bar{l}(-F)|_c) = h^1(c, \bar{l}(-F)|_c)$, because now $c_1(\bar{V}|_b) \neq 0$, nor[13] the vanishing of the H^2-terms in the long exact sequence (4.11). Nevertheless, from the assumption $\bar{\alpha} > 0$ and from $\bar{\beta} < 0$, one has the vanishing of the first two terms such that still $H^0(c, \bar{l}(-F)|_c) \cong \mathrm{Ker}\, \bar{\iota}_1$:

$$0 \to H^0\Big(c, \bar{l}(-F)|_c\Big) \to H^1\Big(\mathcal{E}, \bar{l}(-F-c)\Big) \xrightarrow{\bar{\iota}_1} H^1\Big(\mathcal{E}, \bar{l}(-F)\Big) \to H^1\Big(c, \bar{l}(-F)|_c\Big) \to \dots \quad (6.1)$$

One now wants to find cases where $\bar{\iota}_1$ is a map between spaces of equal dimension[14]

$$h^1\Big(\mathcal{E}, \bar{l}(-F-c)\Big) = h^1\Big(\mathcal{E}, \bar{l}(-F)\Big) \qquad \text{(equality condition)} \qquad (6.2)$$

We proceed now by distinguishing the cases[15] $\bar{\alpha} > n$ and $\bar{\alpha} = n$.

6.1.1 The equality condition in the case $\bar{\alpha} > n$

To compute the difference of the sides of (6.2), one can use formula (C.14) for $\bar{\alpha}, \bar{\beta}$ *and*, in this case of $\bar{\alpha} > n$, *also for* $\bar{\alpha} - n, \bar{\beta} - r$ and gets

$$h^1\left(\mathcal{E}, \bar{l}(-F-c)\right) - h^1\left(\mathcal{E}, \bar{l}(-F)\right) = \left(\bar{\alpha} - \frac{n}{2}\right)(r - n\chi) + \left(\beta + 1 - \frac{r+\chi}{2}\right) n \quad (6.3)$$

Equivalently, with $h^2(\mathcal{E}, \bar{l}(-F-c)) = 0$, one can use the index formula to compute

$$\begin{aligned} h^1\left(\mathcal{E}, \bar{l}(-F-c)\right) - h^1\left(\mathcal{E}, \bar{l}(-F)\right) &= h^0(c, \bar{l}(-F)|_c) - h^1(c, \bar{l}(-F)|_c) \\ &= \deg \bar{l}(-F)|_c - \deg K_c^{1/2} \\ &= \left(\bar{\alpha}s + \bar{\beta}F - \frac{1}{2}(ns + rF + (\chi - 2)F)\right)(ns + rF) \\ &= \left(\bar{\alpha} - \frac{n}{2}\right)(r - n\chi) + \left(\bar{\beta} + 1 - \frac{r+\chi}{2}\right) n \end{aligned} \quad (6.4)$$

Thus one gets vanishing just for

$$-(\bar{\beta} + 1) = \frac{\bar{\alpha} - n}{n}(r - n\chi) - \frac{n+1}{2}\chi \quad (6.5)$$

Note that this is just again the condition (4.22) (i.e., one has (6.2) if *and only if* $c_1(\bar{V}|_{\mathcal{E}}) = 0$, expressed by the relation (4.22) between $\bar{\alpha}$ and $\bar{\beta}$, in this case). The integrality requirement means for n odd (or equally for n even and χ even) that $n|\bar{\alpha}r$ while for $n = 2m$ and χ odd that $\bar{\alpha}r/m$ must be an odd integer.

[13] As $\alpha - n > 0$ from $\alpha = n(\lambda + 1/2), \lambda > 1/2$ does not give necessarily $\bar{\alpha} - n > 0$.

[14] Note that the evaluation by (C.11) also of the lhs of (6.2), which has with $\bar{\alpha} - n$ a smaller s-coefficient in $\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}((\bar{\alpha} - n)s + (\bar{\beta} - r)F)$ which could be potentially ≤ 0, will be appropriate as the condition $\bar{\beta} \leq (\bar{\alpha} - n + 1)\chi + r$ from (C.22) is fulfilled (the rhs is > 0 by (3.11) and $\bar{\alpha} \geq 0$).

[15] Note that $\bar{\alpha} < n$ where $h^1(\mathcal{E}, \bar{l}(-F-c)) = 0$ is not interesting for us as $h^0(c, \bar{l}(-F)|_c) = 0$ would mean that one has no nontrivial section to start with in the procedure (5.3).

6.1.2 The equality condition in the case $\bar{\alpha} = n$

Here one computes $h^1(\mathcal{E}, \mathcal{O}(0s + (\bar{\beta} - r)F)) = -(\bar{\beta}+1) + r$ and gets

$$h^1\Big(\mathcal{E}, \bar{l}(-F-c)\Big) - h^1\Big(\mathcal{E}, \bar{l}(-F)\Big) = -(\bar{\beta}+1) + r + \bar{\alpha}(\bar{\beta}+1) - \left(\frac{\bar{\alpha}(\bar{\alpha}+1)}{2} - 1\right)\chi \quad (6.6)$$

Alternatively one gets this by computing the index on c (using (6.4)) and adding

$$\begin{aligned} h^2\Big(\mathcal{E}, \bar{l}(-F-c)\Big) &= h^0\Big(\mathcal{E}, K_{\mathcal{E}} \otimes \mathcal{O}_{\mathcal{E}}(c) \otimes (\bar{l}(-F))^{-1}\Big) = h^0\Big(b, \mathcal{O}_b(\chi - 2 + r - \bar{\beta})\Big) \\ &= -(\bar{\beta}+1) + r + \chi \end{aligned} \quad (6.7)$$

Thus one gets vanishing just for

$$-(\bar{\beta}+1) = \frac{r}{n-1} - \left(\frac{n}{2}+1\right)\chi \quad (6.8)$$

As this expression has to be integral one notes that for n or χ even one needs to have $r = (n-1)\rho$ with ρ a positive integer; if $n = 2m+1$ and χ are odd one needs to have $\frac{r}{2m} - \frac{1}{2} \in \mathbf{Z}$, i.e. that r/m is an odd integer (for example r must be odd for $n = 3$ with χ odd).

6.2 Interpretation of the equality and reduction condition

We rephrase the condition (6.2) for reduction, given in the numerical parameters above, in a more geometric form by restricting the discussion of the line bundles from $\mathcal{E}$ to c.

6.2.1 The conditions in the case $\bar{\alpha} > n$

The index computation (6.4) shows that the *equality condition* amounts to

$$\bar{l}(-F)|_c = K_c^{1/2} \otimes \bar{\mathcal{F}} \quad (6.9)$$

with $\bar{\mathcal{F}}$ a flat bundle on c playing the same role for $\bar{l}(-F)|_c$ as does $\mathcal{F}$ for $l(-F)|_c$, cf. (3.26). $\bar{\mathcal{F}}$ being a flat bundle this is just an equation of degrees: $\deg \bar{l}(-F) = \deg K_c^{1/2} = \deg l(-F)$.

Now the precise *reduction condition* (5.1) implies, together with (6.9) and (3.26), the necessary condition $p \deg K_c^{1/2} \leq \deg K_c^{1/2}$. So for $p \geq 2$ one gets the condition $\deg K_c \leq 0$ which in view of (3.15) comes down to $r - \frac{n-1}{2}\chi - 1 \leq 0$ (compare the reasoning in appendix F). One gets, in view of (3.12), that $r = 1, \chi = 0$, i.e. the spectral curve $c = C \cap \mathcal{E}$ is an elliptic curve.

6.2.2 The conditions in the case $\bar{\alpha} = n$

Here the *equality condition* amounts to

$$\bar{l}(-F)|_c = K_c^{1/2} \otimes (\omega_{c/b}^{1/2})^{-\frac{1}{n-1}} \otimes \bar{\mathcal{F}} \quad (6.10)$$

where $\omega_{c/b} = K_c \otimes \pi_c^* K_b^{-1}$ is the relative dualizing sheaf with $\deg \omega_{c/b}^{1/2} = n(r - \frac{n-1}{2}\chi)$. Although written, for analogy with (6.9), with the $(n-1)^{th}$ root of the line bundle $\omega_{c/b}^{1/2}$ this is essentially just meant as an equation of degrees (cf. also the remarks after (6.8)).

Now the precise *reduction condition* (5.1) gives, together with (6.10) and (3.26), the necessary condition $p \deg K_c^{1/2} - \frac{p}{n-1} \deg \omega_{c/b}^{1/2} \leq \deg K_c^{1/2}$ and so

$$(p-1)\left(r - \frac{n-1}{2}\chi - 1\right) \leq \frac{p}{n-1}\left(r - \frac{n-1}{2}\chi\right) \tag{6.11}$$

or

$$\Big(p(n-2) - (n-1)\Big)\left(r - \frac{n-1}{2}\chi - 1\right) \leq p \tag{6.12}$$

This is usefully applicable for $n = 3, p > 2$ and $n = 4, p \geq 2$, giving all cases in appendix F.2. Actually we will proceed slightly differently in the concrete derivation in appendix F.2 (using $\lambda \geq p - \frac{1}{2}$ in (F.5) one would get back (6.12)).

6.3 All (strong) reduction cases with equality condition

One has always $\lambda > \frac{1}{2}$ and $\lambda \in \frac{1}{2} + \mathbf{Z}$ for n odd, while for n even the case $\lambda \in \frac{1}{2} + \mathbf{Z}$ needs χ even and $\lambda \in \mathbf{Z}$ needs $r - \chi$ even. The condition $\beta < 0$ can, according to (4.28), be rephrased in the parameters as $r - n\chi \geq \frac{(1+n)/2}{\lambda - 1/2}\chi$; furthermore $r - n\chi > 0$ even for $\chi = 0$ by (3.12). We are usually interested mainly (physically) in $n = 2, 3, 4, 5$ and $\chi = 0$ (for illustration) or 1 (b isolated). Now the detailed study in appendix F gives the following.

6.3.1 Cases with $\bar{\alpha} > n$

For SU(n) bundles one has for $\chi = 0, r = 1, \lambda + \frac{1}{2} \in p\mathbf{Z}^{>1}$ just the p-case

$$l(-F) = \mathcal{O}_{\mathcal{E}}\left(n\left(\lambda + \frac{1}{2}\right)s - \left(\lambda + \frac{1}{2}\right)F\right) \tag{6.13}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\left(n\frac{\lambda + \frac{1}{2}}{p}s - \frac{\lambda + \frac{1}{2}}{p}F\right) \tag{6.14}$$

Now $\chi = 0, r = 1$ were just the conditions for an elliptic c, cf. (3.15). One gets

$$K_c = \mathcal{O}_{\mathcal{E}}(ns + (r + \chi - 2)F)|_c = \mathcal{O}_{\mathcal{E}}(ns - F)|_c \tag{6.15}$$

$$l(-F) = \mathcal{O}_{\mathcal{E}}(ns - F)^{\otimes(\lambda + \frac{1}{2})}|_c \tag{6.16}$$

So here $K_c \cong \mathcal{O}_c \cong l(-F)|_c$, $h^0(c, l(-F)|_c) = 1 > 0$ and $Pfaff \equiv 0$, cf. Ex. 3 below.

6.3.2 Cases with $\bar{\alpha} = n$

Here one has

$$l(-F) = \mathcal{O}_{\mathcal{E}}\left(n\left(\lambda + \frac{1}{2}\right)s + \beta F\right), \qquad \beta = -\left(\left(\lambda - \frac{1}{2}\right)r - \left(n\lambda + \frac{1}{2}\right)\chi + 1\right) \tag{6.17}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\left(ns + \bar{\beta}F\right), \qquad \bar{\beta} = -\left(\frac{1}{n-1}r - \left(\frac{n}{2} + 1\right)\chi + 1\right) \tag{6.18}$$

The (χ, r, λ, p)-list of occurring cases for SU(n) bundles is given in appendix F.2. Furthermore

$$\begin{aligned} \deg\det\iota_1 &= h^1(\mathcal{E}, l(-F)) = -\alpha(\beta+1) + \left(\frac{\alpha(\alpha+1)}{2} - 1\right)\chi \\ &= n\left(\lambda^2 - \frac{1}{4}\right)\left(r - \frac{n}{2}\chi\right) - \chi \qquad (6.19) \\ \deg\det\bar{\iota}_1 &= h^1(\mathcal{E}, \bar{l}(-F)) = -(\bar{\beta}+1) + r = \frac{n}{n-1}r - \left(\frac{n}{2}+1\right)\chi \\ &= \frac{r}{n-1} + \left(r - \frac{n}{2}\chi\right) - \chi \qquad (6.20) \end{aligned}$$

7 The vanishing condition (including example 3)

When considering in section 6.3.1 the reduction case with $\bar{\alpha} > n$ we saw that for $\chi = 0, r = 1, \lambda \in \frac{1}{2} + \mathbf{Z}$ with c elliptic one gets $Pfaff \equiv 0$ by directly computing $h^0(c, l(-F)|_c) = 1 > 0$. In that case both, $l(-F)|_c$ and $\bar{l}(-F)|_c$, were powers of the trivial bundle K_c.

This type of argument can be applied more widely. Note that always $\dim\ker\bar{\iota}_1 \leq h^0(c, \bar{l}(-F)|_c)$ by the analog of the long exact sequence (4.11) for $\bar{l}$ (with equality if $h^0(\mathcal{E}, \bar{l}(-F)) = 0$). Therefore, even is one is not in an equality case where (6.2) holds (when vanishing of $h^0(c, \bar{l}(-F)|_c) = \dim\ker\bar{\iota}_1$ will be controlled by $\det\bar{\iota}_1$) one gets

$$\text{(vanishing condition)}\;\; h^1\Big(\mathcal{E}, \bar{l}(-F-c)\Big) - h^1\Big(\mathcal{E}, \bar{l}(-F)\Big) > 0 \Longrightarrow h^0\Big(c, \bar{l}(-F)|_c\Big) > 0 \quad (7.1)$$

If the reduction condition holds for p then the argument will be completed in the usual way by noting that the existence of $s \in H^0(c, \bar{l}(-F)|_c), s \neq 0$ implies $s^p t \in H^0(c, l(-F)|_c)$; therefore if the (moduli-independent) vanishing condition holds then $Pfaff \equiv 0$.

Let us see how this condition can be applied. If one is in the case $\bar{\alpha} > n$ one has $h^1(\mathcal{E}, \bar{l}(-F-c)) - h^1(\mathcal{E}, \bar{l}(-F)) = \deg\bar{l}(-F)|_c - \deg K_c^{1/2}$ which can not be positive if we want at the same time the necessary condition for reduction $\deg\bar{l}(-F)|_c \leq \deg l(-F)|_c = \deg K_c^{1/2}$ to be fulfilled. Therefore $\bar{\alpha} = n$, i.e. $\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}(ns + \bar{\beta}F)$ and we have

$$\begin{aligned} h^1\Big(\mathcal{E}, \bar{l}(-F-c)\Big) - h^1\Big(\mathcal{E}, \bar{l}(-F)\Big) &= (n-1)(\bar{\beta}+1) + r - \left(\frac{n(n+1)}{2} - 1\right)\chi \\ &= (n-1)\Big(\bar{\beta} + 1 - \frac{n+2}{2}\chi\Big) + r \qquad (7.2) \end{aligned}$$

For SU(3) or SU(4) bundles, the only new case (besides the above mentioned $\chi = 0, r = 1$ case) where this is strictly positive is $n = 3, \lambda = 3/2$ with $r \not\equiv \chi(2)$, cf. appendix F.3.

Let us take a closer look on the case of SU(3) bundles with $\lambda = 3/2$ as this will be an important case in section 9. Here one has $l(-F) = \mathcal{O}_{\mathcal{E}}(6s - (r - 5\chi + 1)F)$ where $\beta < 0$ means $r \geq 5\chi$. Let us take

$$\bar{l}(-F) = \begin{cases} \mathcal{O}_{\mathcal{E}}\Big(3s - (\frac{r-5\chi}{2} + 1)F\Big) & \text{if } r \equiv \chi(2) \\ \mathcal{O}_{\mathcal{E}}\Big(3s - \frac{r-5\chi+1}{2}F\Big) & \text{if } r \not\equiv \chi(2) \end{cases} \qquad (7.3)$$

If $r \equiv \chi(2)$ one gets a moduli-dependent statement, cf. section 9.2.

By contrast, if $r \not\equiv \chi(2)$ one gets the moduli-independent statement $Pfaff \equiv 0$ as here (7.2) becomes just 1 (alternatively one can also argue directly that $h^0(c, \bar{l}(-F)|_c) = \frac{3}{2}(r-\chi-1) - 3(r-\chi-1) + h^0(c, \mathcal{O}_\mathcal{E}((r+\chi-2+\frac{r-5\chi+1}{2})F)|_c) \geq 1$). For $\chi = 0, r = 1$ cf. example 3. For an algebraic vanishing argument in the somewhat special $\chi = 0$ case cf. appendix H.

8 On the difference between the strong and the precise reduction condition

Let us now come back to the precise version of the reduction condition on c instead of the strong version on $\mathcal{E}$. The latter had with (5.9) a precise numerical expession whereas (5.1) implies only the necessary condition for the degrees

$$p(\bar{\alpha}s + \bar{\beta}F)(ns + rF) \leq (\alpha s + \beta F)(ns + rF) \tag{8.1}$$

or explicitly[16]

$$p\Big(\bar{\alpha}(r - n\chi) + n\bar{\beta}\Big) \leq n\left(r - \frac{n-1}{2}\chi - 1\right) \tag{8.2}$$

So the logical relations are

$$\textit{strong reduction (eq. (5.9))} \overset{i)}{\Longrightarrow} \textit{precise reduction (eq. (5.1))} \overset{ii)}{\Longrightarrow} \textit{eq. (8.1)} \tag{8.3}$$

From the onesided implications two questions arise.

i) How much widened is the domain of possible $\bar{l}(-F)$'s by considering the precise condition (5.1) instead of the sharpened condition (5.9)?

ii) The question about sufficiency of the condition (8.1) for the precise reduction (5.1): when is $D - p\bar{D}$ on c (cf. (5.2)), which has degree ≥ 0 by (8.1), actually even effective?

Also in connection with these questions we notice the following.

ad i) Note that if the strong reduction condition is violated we *need actually to check explicitely* that one has $h^0(c, \bar{l}(-F)|_c) > 0$ for which we argued only under certain assumptions in the remark before section 6.1; for example $\bar{\beta} < 0$ followed only because (5.9). 16

ad ii) *If the necessary condition (8.1) for precise reduction is actually saturated* then $t \in H^0(c, \tilde{\mathcal{F}})$, cf. remark after (5.7), is constant ($\tilde{\mathcal{F}}$ is flat and then even trivial; a degree zero effective divisor on c is zero); so, then reduction holds just if $l(-F)|_c \cong \bar{l}(-F)^{\otimes p}|_c$.

[16] Note that we always work under the assumption $\bar{\alpha} \geq n$, cf. footnote 15; here the case $\bar{\alpha} > n$, which implies that the rhs of (8.2) is ≤ 0 (cf. section 6.2.1), still gives $\bar{\beta} \leq 0$ and even < 0 by (3.12); however a case $\bar{\alpha} = n, \bar{\beta} \geq 0$ is possible and will become relevant in section 8.1 and in the example 1 in section 9.3.

8.1 The exceptional case $\bar{\beta} = 0$

We remarked above that $\bar{\beta} < 0$ is necessary to fulfill (5.8), i.e. concretely $p\bar{\beta} \leq \beta$. However, as pointed out, this condition is unnecessarily sharp. Actually one wants only (5.1). For this (5.9) is not necessary (so $\bar{\beta}$ might be ≥ 0; cf. footnote 16); necessary is (8.1).

Let us try to allow $\bar{\beta} = 0$ (this will be relevant for example 1 in section 9.3); consider the first two terms[17] in the long exact sequence (4.11) for $\bar{l}(-F)$: although still $h^0(\mathcal{E}, \mathcal{O}_\mathcal{E}((\bar{\alpha} - n)s - rF)) = 0$ for the first term, one has $h^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(\bar{\alpha}s)) = 1$ for the second one if $\chi > 0$ and $= \bar{\alpha}$ if $\chi = 0$. Thus $\dim \ker \bar{\iota}_1$ equals $h^0(c, \bar{l}(-F))$ *minus* 1 or *minus* $\bar{\alpha}$ if $\chi > 0$ or $= 0$, resp.; so the vanishing of $\det \bar{\iota}_1$ (at a point in the moduli space $\mathcal{M}_\mathcal{E}(c)$) still indicates the existence of a nontrivial section of $\bar{l}(-F)|_c$ and one still has $\det \bar{\iota}_1 | \det \iota_1$.

Let us give the equality condition for the case $\bar{\beta} = 0$. Then one has $h^1(\mathcal{E}, \bar{l}(-F)) = -(\bar{\alpha} - 1)(\bar{\beta} + 1) + (\frac{\bar{\alpha}(\bar{\alpha}+1)}{2} - 1)\chi$ with the remark after (C.14) and gets $-(\bar{\beta} + 1) = \frac{r}{n-2} - \frac{(n+2)(n-1)}{2(n-2)}\chi$ which determines r (where (3.12) excludes $n = 2$ and (3.11) $n > 2, \chi = 0$)

$$r - n\chi = (n-2)\left(\frac{n+1}{2}\chi - 1\right) \tag{8.4}$$

Here, with r assumed fixed by (8.4), the interpretation as in section 6.2 amounts to

$$\bar{l}(-F)|_c = K_c^{1/2} \otimes (K_c^{1/2})^{-\frac{1}{n-1}} \otimes \bar{\mathcal{F}} \tag{8.5}$$

For $n = 3$, where previously $\beta < 0 \Leftrightarrow r \geq 5\chi$, we get $\bar{\beta} = 0 \Leftrightarrow r = 5\chi - 1$, cf. example 1.

This exceptional bundle has the degree on c (using in both cases the r-fix (8.4))

$$\deg(\bar{l}(-F))|_c = \deg \mathcal{O}_\mathcal{E}(ns - 0F)|_c = n(r - n\chi) = n(n-2)\left(\frac{n+1}{2}\chi - 1\right) \tag{8.6}$$

$$\deg K_c^{1/2} = n(n-1)\left(\frac{n+1}{2}\chi - 1\right) \tag{8.7}$$

Therefore, although the conditions (5.9) for strong reduction (5.8) are violated here as $p0 \not\leq \beta < 0$, the necessary condition (8.1) for the precise reduction condition (5.1) is fulfilled if $p(n-2) \leq n - 1$, i.e. for $n \leq 1 + \frac{p}{p-1}$. This holds for $n = 2$ or for $n = 3, p = 2$ where (8.1) is even saturated (and of course trivially for $p = 1$). When is not only the necessary condition (8.1) for precise reduction but even the precise reduction condition (5.1) itself fulfilled? Consider the case $n = 3, p = 2$ where

$$l(-F) = \mathcal{O}_\mathcal{E}\left(3\left(\lambda + \frac{1}{2}\right)s - \left(\lambda - \frac{3}{2}\right)(2\chi - 1)F\right) \tag{8.8}$$

$$\bar{l}(-F) = \mathcal{O}_\mathcal{E}(3s - 0F) \tag{8.9}$$

Note that $l_{\lambda=3/2}(-F) = \bar{l}(-F)^{\otimes 2}$, and so a fortiori $l_{\lambda=3/2}(-F)|_c = \bar{l}(-F)^{\otimes 2}|_c$, but the necessary condition $\beta < 0$ from (4.27) requires here $\lambda > 3/2$, cf. example 1 in section 9. But

[17] Concerning the H^2-terms the second one vanishes by (C.21) as $\bar{\alpha} > 0$ but $h^2(\mathcal{E}, \mathcal{O}_\mathcal{E}((\bar{\alpha} - n)s - rF)) = 0$ only if $\bar{\alpha} > n$ whereas one gets for $\bar{\alpha} = n$ that $h^0(b, \mathcal{O}_b(r + \chi - 2)) = r + \chi - 1$ if not $\chi = 0, r = 1$.

according to the remark before section 8.1 here reduction does hold precisely if $l_\lambda(-F)|_c = \bar{l}(-F)^{\otimes 2}|_c$; so reduction holds just if $l_\lambda(-F)|_c = l_{\lambda=3/2}(-F)|_c$ and one needs to know the sublocus in the moduli space $\mathcal{M}_\mathcal{E}(c)$ where (the isomorphism class of) $l(-F)|_c$ is actually independent of λ, i.e. where $\Lambda = \mathcal{O}_\mathcal{E}(ns-(r-n\chi)F)|_c$ becomes trivial; now cf. section 9.3.

9 Some examples of SU(3) bundles

9.1 Overview over the three main examples

We illustrate the theory with some examples (for example 3 of $Pfaff = 0$ cf. section 7)

	Ex 1	Ex 2	Ex 4
χ	1	1	0
λ	5/2	3/2	3/2
c	$3s+4F$	$3s+5F$	$3s+2F$
K_c	$\mathcal{O}_\mathcal{E}(3s+3F)\|_c$	$\mathcal{O}_\mathcal{E}(3s+4F)\|_c$	$\mathcal{O}_\mathcal{E}(3s)\|_c$
Λ	$\mathcal{O}(3s-F)$	$\mathcal{O}(3s-2F)$	$\mathcal{O}(3s-2F)$
codim Σ_Λ	1	2	1
$l(-F)$	$\mathcal{O}(9s-F)$	$\mathcal{O}(6s-F)$	$\mathcal{O}(6s-3F)$
$Pfaff = \det\iota_1$	$f^{11}Q_{11}$	f^4	f^4
$\bar{l}(-F)$	$\mathcal{O}(3s)$	$\mathcal{O}(3s-F)$	$\mathcal{O}(3s-2F)$
$f = \det\bar{\iota}_1$	f_Λ	$Res(B,A)$	f_Λ

For these examples we find that $f := \det\bar{\iota}_1$ equals f_Λ if that relation is possible at all, that is if the codimension of the $f_\Lambda = 0$ locus Σ_Λ is 1, as in examples 1 and 4: in example 4 the naive codimension $r-n\chi$ (cf. section 3.2.2) of Σ_Λ is 2 but $\bar{l}(-F) = \Lambda$ leads to $f = f_\Lambda$; in example 2 the codimension is 2: one has to demand $R_1 = 0$ and $R_2 = 0$, i.e. $(R_1, R_2) = f_\Lambda = (Res(B, A_i))_i$ where $A = \prod_{i=1}^{r-n\chi} A_i$ (cf. footnote 7) and one has then that codim $\Sigma_\Lambda > 1$ (the general case). Although Σ_Λ is a sublocus of $Pfaff = 0$ (cf. the remark after (4.35)) the 'scalar function' dividing $Pfaff$ is $Res(B,A)$, cf. footnote 9.

9.2 Bundles with $\lambda = 3/2$

Here one has $l(-F) = \mathcal{O}_\mathcal{E}(6s-(r-5\chi+1)F)$ where $\beta < 0$ means $r \geq 5\chi$. Let us take

$$\bar{l}(-F) = \begin{cases} \mathcal{O}_\mathcal{E}\Big(3s - (\frac{r-5\chi}{2}+1)F\Big) & \text{if } r \equiv \chi(2) \\ \mathcal{O}_\mathcal{E}\Big(3s - \frac{r-5\chi+1}{2}F\Big) & \text{if } r \not\equiv \chi(2) \end{cases} \tag{9.1}$$

For $r \not\equiv \chi(2)$ we found $Pfaff \equiv 0$ in section 7. For $r \equiv \chi(2)$ one gets a moduli-dependent statement: if an $\rho \in H^0(c, \bar{l}(-F)|_c), \rho \neq 0$ exists (what is described by $\det\bar{\iota}_1 = 0$) then $\rho^2 u$ and $\rho^2 v$ are non-trivial elements of $H^0(c, l(-F)|_c)$, what suggests for this case that $Pfaff = f^2 g$. Note that at most $Pfaff = f^4$ as $\deg Pfaff = 6r - 10\chi, \deg f = \frac{3r-5\chi}{2}$.

9.2.1 The case $\chi \geq 1$ (including example 2 as minimal r-value)

For $r = 5\chi$, where $\deg Pfaff = 4r, \deg f = r$ and $\chi \geq 1$ by (3.12), one gets

$$l(-F) = \mathcal{O}_{\mathcal{E}}(6s - F) \tag{9.2}$$
$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}(3s - F) \tag{9.3}$$

One then has also $\rho z \in H^0(c, l(-F)|_c)$ such that possibly $Pfaff = f^3 h$ in this case.

For $\chi = 1$, where $c = 3s + 5F$, this is realized in example 2. Note that there the matrix induced by $\bar{\iota}$ is $m_5 = \mathcal{D}_5$ in (G.13), such that $f = \det \bar{\iota}_1 = Res(A_2, B_3)$, cf. section I.1.

This is case $3^{\chi=1}$ of appendix F.2. According to (4.30) one has then to consider the map $\iota_1 : H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s + (\beta - r)F)\Big) \longrightarrow H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(6s + \beta F)\Big)$ between spaces of dimension $6r - 10\chi$ (for concrete matrix representations cf. appendix G) This map is induced by multiplication with an element $\tilde{\iota} \in H^0\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s + rF)\big)$. The determinant of the ι_1 for $r \equiv \chi(2)$ has as factor the determinant of a corresponding map $i_r : i_r : H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}\big(0s + (\bar{\beta} - r)F\big)\Big) \longrightarrow H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}\big(3s + \bar{\beta}F\big)\Big)$ (again induced by multiplication with $\tilde{\iota} \in H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s + rF)\Big)$, so depending on the same moduli) between two spaces of dimension $\frac{1}{4}(6r - 10\chi)$.

Now, what was generically $s|_c = \{p_0, t_1\} + \{p_0, t_2\}$, $F_{t_i}|_c = c_{t_i} = \{p_0 + p_i^+ + p_i^-, t_i\}$ (with $A_2(t_i) = 0$, $i = 1, 2$, cf. (3.20)) changes at the specialisation locus $f = \det \bar{\iota}_1 = \det \mathcal{D}_5 = Res(A_2, B_3) = 0$ (where $B_3(t_1) = 0$; the non-generic case for (3.20)): the (w) line becomes, in the $\mathbf{P}^2_{t_1}$-fibre of $\mathcal{W}_b$, the (z) line and p_0 becomes a three-fold point of c_{t_1}

$$(z)|_c = 3s|_c = \{3p_0, t_1\} + \{3p_0, t_2\} = F_{t_1}|_c + \{3p_0, t_2\} \neq F_{t_1}|_c + F_{t_2}|_c \tag{9.4}$$

In other words, in the specialisation locus $f = 0$ one gets $(3s - F)|_c \sim$ effective or equivalently $h^0(c, \mathcal{O}(3s - F)|_c) \geq 1$ (the existence of $\rho \neq 0$); therefore also $(6s - F)|_c \sim$ effective or $h^0(c, l(-F)|_c) \geq 1$; in other words $f \,|\, Pfaff$.

To demand that even $(3s - 2F)|_c \sim$ effective is a more restrictive condition, i.e. $R_1 = 0 = R_2$ is a proper sublocus of $f = 0$ (so here $f_\Lambda = (R_1, R_2)$, cf. footnote 7). Note that $(3s - 2F)|_c \sim$ effective $\Longrightarrow (3s - F)|_c \sim$ effective; so, if[18] $R_1 = 0 = R_0$ at certain points in moduli space, then $f = 0$ there, as becomes manifest in the representation

$$f = \frac{1}{a_2^2} \det \begin{pmatrix} a_0 & a_1 & a_2 \\ R_1 & R_0 & 0 \\ 0 & R_1 & R_0 \end{pmatrix} \tag{9.5}$$

9.2.2 The case $\chi = 0$ (including example 4 as minimal r-value)

For $\chi = 0$ one has r even and $\deg Pfaff = 12\frac{r}{2}$ and $\deg f = 3\frac{r}{2}$. The minimal r-value (which must be > 0) is 2, cf. example 4, where $c = 3s + 2F$ and

$$l(-F) = \mathcal{O}_{\mathcal{E}}(6s - 3F) \tag{9.6}$$
$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}(3s - 2F) = \Lambda \tag{9.7}$$

[18] Cf. section 3.2.2, we call here $R_2 = R_0$; cf. also appendix I.2.

Precisely on the specialisation locus $f = \det \bar{\iota}_1 = \det \mathcal{D}_3^{\chi=0} = 0$, cf. (G.15), the degree 0 line bundle $\bar{l}(-F)|_c$ gets a nonzero section and so becomes trivial, in other words

$$f_\Lambda = f \tag{9.8}$$

Remark.

i) Following strictly the procedure of example 2 one would expect codim $\Sigma_\Lambda = 2$ and f_Λ a two-component expression: the condition $A_2|B_2$ is here[19] $M_0^c = 0 = M_1^c$ while

$$Res(B_2, A_2) = \det \begin{pmatrix} b_2 & b_1 & b_0 & 0 \\ 0 & b_2 & b_1 & b_0 \\ a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \end{pmatrix} = \det \begin{pmatrix} M_1^c & M_0^c \\ M_2^c & M_0^c \end{pmatrix} \tag{9.9}$$

(note that $0 = a_0 M_0^c - a_1 M_1^c + a_2 M_2^c$ allows to eliminate M_2^c). Now one has

$$f = \frac{1}{a_2} \det \begin{pmatrix} M_1^c & M_0^c \\ M_1^b & M_0^b \end{pmatrix} \tag{9.10}$$

such that the locus where $M_0^c = 0 = M_1^c$ is certainly a subset of the $f = 0$ locus, cf. the remark after (3.22). But $A_2|B_2$ is only a sufficient, not a necessary condition for $\Lambda|_c \cong \mathcal{O}_c$, cf. (3.22). Here, in this case of symmetry between C, B and A because of $\chi = 0$, the precise condition for $\Lambda|_c \cong \mathcal{O}_c$ turns out to be just linear dependence among C, B and A which represents the *single* condition $f = 0$; this comes as here $\Lambda = \bar{l}(-F)$ and demanding a nontrivial section over c leads to the determinantal condition $f = 0$.

ii) That $3s|_c \sim 2F|_c$ on the locus $f = 0$ can also be seen directly from $\det \bar{\iota}_1 = \det \mathcal{D}_3^{\chi=0} = 0$, cf. (G.15), instead of arguing via the long exact sequence. The equation for the fibre of c over $t_1 = u/v \in b$, say, is $C_2(t_1)z + B_2(t_1)x + A_2(t_1)y = 0$ and these three points lie also in the divisor $\alpha z + \beta x + \gamma y = 0$, understood as $3s|_c$, where $\alpha = C_2(t_1), \beta = B_2(t_1), \gamma = A_2(t_1)$; this divisor of degree 6 contains three further points; we want to see that these can arise as a further fibre triplet over t_2, say. So we ask whether $t_1, t_2 (\neq t_1) \in b$ exist such that the 3-vectors $(D_2(t_1))$ and $(D_2(t_2))$ where $D_2 = C_2, B_2, A_2$ are linearly dependent, i.e. whether $k \in \mathbf{C}^*$ exists such that $D_2(t_2) = k D_2(t_1)$ for $D_2 = C_2, B_2, A_2$ which indeed just amounts to the nontrivial solvability of $\mathcal{D}_3^{\chi=0} \cdot (kt_1^2 - t_2^2, kt_1 - t_2, k-1)^t = 0$.

9.3 Example 1 with $\lambda = 5/2$

An SU(3) bundle, $\chi = 1$ with $c = 3s + 4F$, so[20] $l(-F) = \mathcal{O}_\mathcal{E}\big(9s - F\big)$, gives a map

$$\iota_1 : H^1\Big(\mathcal{E}, \mathcal{O}_\mathcal{E}\big(6s - 5F\big)\Big) \longrightarrow H^1\Big(\mathcal{E}, \mathcal{O}_\mathcal{E}\big(9s - F\big)\Big) \tag{9.11}$$

[19] With minors w.r.t. a development of (G.15) w.r.t. the c-row, and in (9.10) also w.r.t. the b-row.

[20] This is reflected in the sense of appendix E from an example 1 with $\lambda = -5/2$.

between 44-dimensional spaces by (C.14). ι_1 is induced from multiplication with $\tilde{\iota} = Cz + Bx + Ay$ where (for the notation cf. appendix G)

$$\tilde{\iota} \in H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s + rF)\Big) = \bigoplus_{w' \in StB(\Sigma)} w'\, H^0\Big(b, r - [w']\,\chi\Big) = \bigoplus_{w' \in StB(\Sigma)} w'\, S^{r-[w']\,\chi}\, V \quad (9.12)$$

(i.e. $w' \in H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s + [w']\chi F))$, cf. (G.4)) with the accompanying coefficients

$$\begin{aligned} C &= c_4u^4 + c_3u^3v + c_2u^2v^2 + c_1uv^3 + c_0v^4 && \in S^r\, V = S^4\, V = Hom(S^4\, V^*, \mathbf{C}) && (9.13)\\ B &= b_2u^2 + b_1uv + b_0v^2 && \in S^{r-2\chi}\, V = S^2\, V = Hom(S^2\, V^*, \mathbf{C}) && (9.14)\\ A &= a_1u + a_0v && \in S^{r-3\chi}\, V = V = Hom(V^*, \mathbf{C}) && (9.15) \end{aligned}$$

With $l(-F)$ being not among the strong reduction cases we adopt the more general approach of section 8.1: we use $\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}(3s - 0F)$ and get the map

$$\bar{\iota}_1 : H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(0s - 4F)\Big) \longrightarrow H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s - 0F)\Big) \quad (9.16)$$

between 3-dimensinal spaces. Or, in matrix form, (having $\bar{\beta} = 0$ the first line is absent)

$$m_r \; : \qquad \begin{pmatrix} C \odot S^{-\bar{\beta}-2} \quad\;\; V^* \\ B \odot S^{-\bar{\beta}-2+2\chi}\, V^* \\ A \odot S^{-\bar{\beta}-2+3\chi}\, V^* \end{pmatrix} \quad (9.17)$$

Concretely this means one builds the map

$$\mathcal{D}_3 := B \,\oplus\, A \underline{\odot}\, V^* = \begin{pmatrix} b_2 & b_1 & b_0 \\ a_1 & a_0 & 0 \\ 0 & a_1 & a_0 \end{pmatrix} \;\in\; Hom(S^2\, V^*, \mathbf{C} \oplus V^*) \quad (9.18)$$

One finds (where $\det \mathcal{D} = Res(A_1, B_2)$, cf. section A.1 and I.1)

$$\det\, \iota_1 = -4^{10} \,\Big(\det\, \mathcal{D}_3\Big)^{11}\, Q_{11} \quad (9.19)$$

Let us discuss the pair

$$\begin{aligned} l(-F) &= \mathcal{O}_{\mathcal{E}}(9s - F) && (9.20)\\ \bar{l}(-F) &= \mathcal{O}_{\mathcal{E}}(3s - 0F) && (9.21) \end{aligned}$$

from the point of view of the reduction philosophy described in section 8.1. Concerning $\bar{l}(-F)$ we are here in the exceptional case $\bar{\alpha} = n, \bar{\beta} = 0$; so the equality condition leads to the r-fix $r = 4$ from (8.4) and the discussion after (8.9) shows that, although the $\lambda = 3/2$ analogue of $l(-F)$ would have given $l(-F) = \bar{l}(-F)^{\otimes 2}$, here $\lambda \geq 5/2$ is required.

Now the long exact sequence for $\bar{l}$

$$\begin{aligned} 0 \qquad\qquad &\longrightarrow H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s)\Big) \stackrel{\rho}{\longrightarrow} H^0\Big(c, \mathcal{O}_{\mathcal{E}}(3s)|_c\Big)\\ \stackrel{\delta}{\longrightarrow} H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(-4F)\Big) &\stackrel{\bar{\iota}_1}{\longrightarrow} H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s)\Big) \end{aligned} \quad (9.22)$$

gives $\ker \bar{\iota}_1 \cong H^0(c, \mathcal{O}_{\mathcal{E}}(3s)|_c)/\rho H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s))$. So generically, for $f := \det \bar{\iota}_1 \neq 0$, one has $h^0(c, \mathcal{O}_{\mathcal{E}}(3s)|_c) = 1$ from the section given by z. The necessary condition (8.1) for precise reduction $p \deg \bar{l}(-F)|_c \leq \deg l(-F)|_c$ gives $p = 1$ or 2; we consider here the case $p = 2$. Then the bundle $\tilde{\mathcal{F}} = \mathcal{O}_c(D - 2\bar{D})$ from section 8, which needs to have a nontrivial section to carry through the reduction procedure, is the bundle $\Lambda|_c = \mathcal{O}_{\mathcal{E}}(3s - F)|_c$, cf. (3.16) (this is also in line with the remarks after (8.9)); this flat bundle has $h^0(c, \Lambda|_c)$ either 0 or 1. So, if both, $\bar{l}(-F)|_c = \mathcal{O}_{\mathcal{E}}(3s)|_c$ and $\Lambda|_c = \mathcal{O}_{\mathcal{E}}(3s - F)|_c$, have a nontrivial section (the first line bundle has z) then so does $l(-F)|_c$. So (cf. (4.36))

$$f_\Lambda \,|\, Pfaff \tag{9.23}$$

To find the result $f|Pfaff$ we now show that here actually $f = f_\Lambda$.

First we argue for $f_\Lambda|f$. Now, with (3.14), one has

$$h^0(c, \mathcal{O}_{\mathcal{E}}(3s)|_c) = 3 - 6 + h^0(c, \mathcal{O}_{\mathcal{E}}(3F)|_c) =: -3 + P \tag{9.24}$$

$$h^0(c, \mathcal{O}_{\mathcal{E}}(3s - F)|_c) = 0 - 6 + h^0(c, \mathcal{O}_{\mathcal{E}}(4F)|_c) =: -6 + Q \tag{9.25}$$

such that always $P \geq 3, Q \geq 6$. Generically one has $P = 4$, from the section z; note that $P = h^0(c, \mathcal{O}_{\mathcal{E}}(3F)|_c) \geq h^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3F)) = h^0(b, \mathcal{O}_b(3)) = 4$. As remarked above actually $Q = 6$ or 7. By (4.36) the latter is sufficient for reduction (giving (9.23)): $Q = 7 \iff h^0(c, \Lambda|_c) \geq 1 \iff \Lambda|_c \cong \mathcal{O}_c \iff f_\Lambda = 0 \Longrightarrow l(-f)|_c \cong \mathcal{O}_{\mathcal{E}}(2F)|_c$. Note that $Q \geq 7 \Longrightarrow P \geq 5$ as then $\mathcal{O}(3s)|_c \cong \mathcal{O}(F)|_c$ with the two sections $\pi_c^* u$ and $\pi_c^* v$; so one gets another (linearly independent) section z' of $\mathcal{O}_{\mathcal{E}}(3s)|_c$, such that

$$f_\Lambda \,|\, f \tag{9.26}$$

Now we argue for $f|f_\Lambda$: the generic result $s|_c = \{p_0, t_0\}\,,\ F_{t_0}|_c = c_{t_0} = \{p_0 + p_+ + p_-, t_0\}$ (with $A_1(t_0) = 0$, cf. (3.20)) changes at the locus $f = \det \bar{\iota}_1 = \det \mathcal{D}_3 = Res(A_1, B_2) = 0$ (where $B_2(t_0) = 0$): p_0 becomes a three-fold point of c_{t_0}. So one has in $\mathcal{M}_{\mathcal{E}}(c)$

$$f = 0 \iff (z)|_c = 3s|_c = \{3p_0, t_0\} = F_{t_0}|_c \tag{9.27}$$

Therefore $f \,|\, f_\Lambda$ and one gets here that actually even $f = f_\Lambda$ and so $f|Pfaff$ by (9.23).

A The polynomial factors of the examples in detail

After restricting from the twodimensional base B to the rational curve $b \subset B$ (with its own homogeneous coordinates u and v) the defining equation for the spectral curve c (of class $ns + rF$, cf. section 3.2) in the elliptic surface $\mathcal{E}$ over b is

$$w := C_r(u,v)z + B_{r-2\chi}(u,v)x + A_{r-3\chi}(u,v)y = 0 \tag{A.1}$$

Here the subscripts denote the degrees of the homogeneous polynomials and for the cases of $B = \mathbf{F_k}$ (as for the examples in the table of section 1.1) one has $\chi = k - 2$, cf. section 3.1; x, y, z are, of course, the usual Weierstrass coordinates for the elliptic fibre, cf. section 2.1.

A.1 Detailed consideration of example 1

Here the spectral curve has the equation (with $D_m = \sum_{i=0}^{m} d_i u^i v^{m-i}$ for $D = A, B, C$)

$$w = C_4 z + B_2 x + A_1 y = 0 \tag{A.2}$$

and the experimental result is, in more detail, the following

$$Pfaff \;=\; -4^{10}\,\mathcal{D}_3^{11}\,Q_{11} \tag{A.3}$$

where the factors have the following meaning: $\mathcal{D}_3$ is defined by

$$\mathcal{D}_3 \;=\; b_2 a_0^2 - b_1 a_1 a_0 + b_0 a_1^2 \tag{A.4}$$

which is of course the following resultant (cf. appendix I)

$$\mathcal{D}_3 := Res(B_2, A_1) = \det\begin{pmatrix} b_2 & b_1 & b_0 \\ a_1 & a_0 & 0 \\ 0 & a_1 & a_0 \end{pmatrix} \tag{A.5}$$

Q_{11}: The Giant. The structure of the second factor is more involved, it has the following 132 terms

$$\begin{aligned}
Q_{11} \;=\; & -2a_1^6b_0b_1^3c_0 + 2a_0a_1^5b_1^4c_0 + 4a_1^6b_0^2b_1b_2c_0 + 4a_0a_1^5b_0b_1^2b_2c_0 - 10a_0^2a_1^4b_1^3b_2c_0 \\
& -8a_0a_1^5b_0^2b_2^2c_0 + 20a_0^3a_1^3b_1^2b_2^2c_0 - 20a_0^4a_1^2b_1b_2^3c_0 + 8a_0^5a_1b_2^4c_0 + 48a_1^8b_1c_0^2 \\
& -96a_0a_1^7b_2c_0^2 + 2a_1^6b_0^2b_1^2c_1 - 2a_0a_1^5b_0b_1^3c_1 - 2a_1^6b_0^3b_2c_1 - 6a_0a_1^5b_0^2b_1b_2c_1 \\
& +10a_0^2a_1^4b_0b_1^2b_2c_1 + 10a_0^2a_1^4b_0^2b_2^2c_1 - 20a_0^3a_1^3b_0b_1b_2^2c_1 + 10a_0^4a_1^2b_0b_2^3c_1 + 2a_0^5a_1b_1b_2^3c_1 \\
& -2a_0^6b_2^4c_1 - 24a_1^8b_0c_0c_1 - 72a_0a_1^7b_1c_0c_1 + 168a_0^2a_1^6b_2c_0c_1 + 24a_0a_1^7b_0c_1^2 \\
& +24a_0^2a_1^6b_1c_1^2 - 72a_0^3a_1^5b_2c_1^2 - 2a_1^6b_0^3b_1c_2 + 2a_0a_1^5b_0^2b_1^2c_2 + 8a_0a_1^5b_0^3b_2c_2 \\
& -10a_0^2a_1^4b_0^2b_1b_2c_2 + 10a_0^4a_1^2b_0b_1b_2^2c_2 - 2a_0^5a_1b_1^2b_2^2c_2 - 8a_0^5a_1b_0b_2^3c_2 + 2a_0^6b_1b_2^3c_2 \\
& +48a_0a_1^7b_0c_0c_2 + 48a_0^2a_1^6b_1c_0c_2 - 144a_0^3a_1^5b_2c_0c_2 - 72a_0^2a_1^6b_0c_1c_2 - 24a_0^3a_1^5b_1c_1c_2 \\
& +120a_0^4a_1^4b_2c_1c_2 + 48a_0^3a_1^5b_0c_2^2 - 48a_0^5a_1^3b_2c_2^2 + 2a_1^6b_0^4c_3 - 2a_0a_1^5b_0^3b_1c_3 \\
& -10a_0^2a_1^4b_0^3b_2c_3 + 20a_0^3a_1^3b_0^2b_1b_2c_3 - 10a_0^4a_1^2b_0b_1^2b_2c_3 + 2a_0^5a_1b_1^3b_2c_3 - 10a_0^4a_1^2b_0^2b_2^2c_3 \\
& +6a_0^5a_1b_0b_1b_2^2c_3 - 2a_0^6b_1^2b_2^2c_3 + 2a_0^6b_0b_2^3c_3 - 72a_0^2a_1^6b_0c_0c_3 - 24a_0^3a_1^5b_1c_0c_3 \\
& +120a_0^4a_1^4b_2c_0c_3 + 96a_0^3a_1^5b_0c_1c_3 - 96a_0^5a_1^3b_2c_1c_3 - 120a_0^4a_1^4b_0c_2c_3 + 24a_0^5a_1^3b_1c_2c_3 \\
& +72a_0^6a_1^2b_2c_2c_3 + 72a_0^5a_1^3b_0c_3^2 - 24a_0^6a_1^2b_1c_3^2 - 24a_0^7a_1b2c_3^2 - 8a_0a_1^5b_0^4c_4 \\
& +20a_0^2a_1^4b_0^3b_1c_4 - 20a_0^3a_1^3b_0^2b_1^2c_4 + 10a_0^4a_1^2b_0b_1^3c_4 - 2a_0^5a_1b_1^4c_4 - 4a_0^5a_1b_0b_1^2b_2c_4 \\
& +2a_0^6b_1^3b_2c_4 + 8a_0^5a_1b_0^2b_2^2c_4 - 4a_0^6b_0b_1b_2^2c_4 + 96a_0^3a_1^5b_0c_0c_4 - 96a_0^5a_1^3b_2c_0c_4 \\
& -120a_0^4a_1^4b_0c_1c_4 + 24a_0^5a_1^3b_1c_1c_4 + 72a_0^6a_1^2b_2c_1c_4 + 144a_0^5a_1^3b_0c_2c_4 - 48a_0^6a_1^2b_1c_2c_4 \\
& -48a_0^7a_1b_2c_2c_4 - 168a_0^6a_1^2b_0c_3c_4 + 72a_0^7a_1b_1c_3c_4 + 24a_0^8b_2c_3c_4 + 96a_0^7a_1b_0c_4^2 \\
& -48a_0^8b_1c_4^2 - 2a_1^8b_0^2b_1g_0 + 4a_0a_1^7b_0b_1^2g_0 - 2a_0^2a_1^6b_1^3g_0 + 4a_0a_1^7b_0^2b_2g_0 \\
& -12a_0^2a_1^6b_0b_1b_2g_0 + 8a_0^3a_1^5b_1^2b_2g_0 + 8a_0^3a_1^5b_0b_2^2g_0 - 10a_0^4a_1^4b_1b_2^2g_0 + 4a_0^5a_1^3b_2^3g_0 \\
& +a_1^8b_0^3g_1 - a_0a_1^7b_0^2b_1g_1 - a_0^2a_1^6b_0b_1^2g_1 + a_0^3a_1^5b_1^3g_1 - a_0^2a_1^6b_0^2b_2g_1
\end{aligned}$$

$$
\begin{aligned}
&+6a_0^3a_1^5b_0b_1b_2g_1-5a_0^4a_1^4b_1^2b_2g_1-5a_0^4a_1^4b_0b_2^2g_1+7a_0^5a_1^3b_1b_2^2g_1-3a_0^6a_1^2b_2^3g_1\\
&-2a_0a_1^7b_0^3g_2+4a_0^2a_1^6b_0^2b_1g_2-2a_0^3a_1^5b_0b_1^2g_2-2a_0^3a_1^5b_0^2b_2g_2+2a_0^5a_1^3b_1^2b_2g_2\\
&+2a_0^5a_1^3b_0b_2^2g_2-4a_0^6a_1^2b_1b_2^2g_2+2a_0^7a_1b_2^3g_2+3a_0^2a_1^6b_0^3g_3-7a_0^3a_1^5b_0^2b_1g_3\\
&+5a_0^4a_1^4b_0b_1^2g_3-a_0^5a_1^3b_1^3g_3+5a_0^4a_1^4b_0^2b_2g_3-6a_0^5a_1^3b_0b_1b_2g_3+a_0^6a_1^2b_1^2b_2g_3\\
&+a_0^6a_1^2b_0b_2^2g_3+a_0^7a_1b_1b_2^2g_3-a_0^8b_2^3g_3-4a_0^3a_1^5b_0^3g_4+10a_0^4a_1^4b_0^2b_1g_4\\
&-8a_0^5a_1^3b_0b_1^2g_4+2a_0^6a_1^2b_1^3g_4-8a_0^5a_1^3b_0^2b_2g_4+12a_0^6a_1^2b_0b_1b_2g_4-4a_0^7a_1b_1^2b_2g_4\\
&-4a_0^7a_1b_0b_2^2g_4+2a_0^8b_1b_2^2g_4
\end{aligned}
\tag{A.6}
$$

The Giant is friendly. The expression for Q_{11} looks unwieldily. However meditation reveals[21]

$$
Q_{11} = -D_1 + D_2 + D_3 \tag{A.7}
$$

(cf. appendix A.4) where one has the individual terms (with $zy^2 = 4x^3 - G_4xz^2 - G_6z^3$ the elliptic fibre over b; thus here a dependence on the complex structure moduli of X enters)

$$
\text{type }"(ga)b^3a^7" \qquad D_1 = \mathcal{D}_3^2 \cdot \frac{R(G_4, B_2^2, A_1^2)}{\mathcal{D}_3} \tag{A.8}
$$

$$
\text{type }"c^2ba^8" \qquad D_2 = 24\,\mathcal{D}_5 \cdot \frac{R(C_4, B_2^2, A_1^2)}{\mathcal{D}_3} \tag{A.9}
$$

$$
\text{type }"cb^4a^6" \qquad D_3 = 2\,\mathcal{D}_3 \cdot R(C_4, A_1^4, B_2) \tag{A.10}
$$

Here the meaning of the factors is the following: $\mathcal{D}_5$ is defined by

$$
\mathcal{D}_5 = \sum_{i=0}^{4} (-1)^i c_i a_0^i a_1^{4-i} \tag{A.11}
$$

which is of course the following resultant

$$
\mathcal{D}_5 := Res(C_4, A_1) = \det \begin{pmatrix} c_4 & c_3 & c_2 & c_1 & c_0 \\ a_1 & a_0 & 0 & 0 & 0 \\ 0 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & a_1 & a_0 \end{pmatrix} \tag{A.12}
$$

The other terms are the following 'multi-resultants'

$$
R(C_4, B_2^2, A_1^2) = \det \begin{pmatrix} c_4 & c_3 & c_2 & c_1 & c_0 \\ b_2^2 & 2b_2b_1 & 2b_2b_0 + b_1^2 & 2b_1b_0 & b_0^2 \\ a_1^2 & 2a_1a_0 & a_0^2 & 0 & 0 \\ 0 & a_1^2 & 2a_1a_0 & a_0^2 & 0 \\ 0 & 0 & a_1^2 & 2a_1a_0 & a_0^2 \end{pmatrix} \tag{A.13}
$$

[21] As a minor difference we find in contrast to [8] a minus-sign in front of D_1 (although not an overall sign this is tunable by the sign of G_4) and get (A.3) (with a prefactor 4^3) with (A.7) by using elements $z^2x, z^2y, zx^2, zxy, x^3, x^2y, xy^2, y^3$ for the $H^1(\mathcal{E}, \mathcal{O}(9s - F))$ decomposition, cf. also appendix G; note that in line with the treatment for the f-factor would actually be a representation as *one* determinant.

and correspondingly for G_4; realise that this contains actually a $\mathcal{D}_3$-factor ! Finally

$$R(C_4, A_1^4, B_2) = -\det \begin{pmatrix} c_4 & c_3 & c_2 & c_1 & c_0 \\ b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 \\ a_1^4 & 4a_1^3 a_0 & 6a_1^2 a_0^2 & 4a_1 a_0^3 & a_0^4 \end{pmatrix} \tag{A.14}$$

A.2 Detailed consideration of example 2

The spectral curve equation is $w = C_5 z + B_3 x + A_2 y = 0$ and the experimental result is

$$Pfaff = (\mathcal{D}_5^{II})^4 \tag{A.15}$$

where

$$\begin{aligned} \mathcal{D}_5^{II} &= a_2^3 b_0^2 - a_1 a_2^2 b_0 b_1 + a_0 a_2^2 b_1^2 + a_1^2 a_2 b_0 b_2 - 2a_0 a_2^2 b_0 b_2 - a_0 a_1 a_2 b_1 b_2 + a_0^2 a_2 b_2^2 \\ &\quad - a_1^3 b_0 b_3 + 3a_0 a_1 a_2 b_0 b_3 + a_0 a_1^2 b_1 b_3 - 2a_0^2 a_2 b_1 b_3 - a_0^2 a_1 b_2 b_3 + a_0^3 b_3^2 \end{aligned} \tag{A.16}$$

which is the following resultant

$$\mathcal{D}_5^{II} = R(B_3, A_2) = \det \begin{pmatrix} b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \\ a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 \end{pmatrix} \tag{A.17}$$

A.3 Detailed consideration of example 4

The spectral curve equation is $w = C_2 z + B_2 x + A_2 y = 0$ and the experimental result is

$$Pfaff = (\mathcal{D}_3^{IV})^4 \tag{A.18}$$

where

$$\mathcal{D}_3^{IV} = c_2 b_1 a_0 - c_2 b_0 a_1 + c_1 b_0 a_2 - c_1 b_2 a_0 + c_0 b_2 a_1 - c_0 b_1 a_2 \tag{A.19}$$

which is the following 'multi-resultant'

$$\mathcal{D}_3^{IV} = R(C_2, B_2, A_2) = \det \begin{pmatrix} c_2 & c_1 & c_0 \\ b_2 & b_1 & b_0 \\ a_2 & a_1 & a_0 \end{pmatrix} \tag{A.20}$$

A.4 The decomposition of the Giant factor Q_{11}

To follow in greater detail the process how the decomposition of Q_{11} arises note that in the expression for Q_{11} three types of terms occur: first those with g_i (and with no c_i), second

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those with c_ic_j and third those with c_i. More precisely the first group of 46 terms has the *type* "$(ga)b^3a^7$"

$$\begin{aligned}
D_1 =\ & g_0a_0\Big(10a_1^2b_1^2b_2+10b_2^2(a_1^2b_0-2a_0a_1b_1+a_0^2b_2)\Big)a_0^2a_1^3\\
&+(g_0a_1+g_1a_0)\Big(-2a_1^5b_0^2b_1-2a_0^2a_1^3b_1^3-12a_0^2a_1^3b_0b_1b_2+10a_0^4a_1b_1b_2^2-6a_0^5b_2^3\\
&\qquad+4a_0a_1^4b_0(b_1^2+b_0b_2)-2a_0^3a_1^2b_2(b_1^2+b_0b_2)\Big)a_1^2\\
&+(g_2a_0+g_1a_1)\Big(a_1^6b_0^3+a_0a_1^5b_0^2b_1+3a_0^3a_1^3b_1^3+18a_0^3a_1^3b_0b_1b_2-3a_0^5a_1b_1b_2^2\\
&\qquad+3a_0^6b_2^3-5a_0^2a_1^4b_0(b_1^2+b_0b_2)-3a_0^4a_1^2b_2(b_1^2+b_0b_2)\Big)a_1\\
&+(g_3a_0+g_2a_1)\Big(-3a_1^6b_0^3+3a_0a_1^5b_0^2b_1-3a_0^3a_1^3b_1^3-18a_0^3a_1^3b_0b_1b_2-a_0^5a_1b_1b_2^2\\
&\qquad-a_0^6b_2^3+(3a_0^2a_1^4b_0+5a_0^4a_1^2b_2)(b_1^2+b_0b_2)\Big)a_0\\
&+(g_4a_0+g_3a_1)\Big(6a_1^5b_0^3-10a_0a_1^4b_0^2b_1+2a_0^3a_1^2b_1^3+12a_0^3a_1^2b_0b_1b_2+2a_0^5b_1b_2^2\\
&\qquad+2a_0^2a_1^3b_0(b_1^2+b_0b_2)-4a_0^4a_1b_2(b_1^2+b_0b_2)\Big)a_0^2\\
&+g_4a_1\Big(-10a_0^2b_0b_1^2-10b_0^2(a_1^2b_0-2a_0a_1b_1+a_0^2b_2)\Big)a_0^3a_1^2
\end{aligned}\tag{A.21}$$

(with terms already regrouped according to the coefficients not of G_4 but of G_4A_1).

The second group of 40 terms has the *type* "c^2ba^8"

$$\begin{aligned}
D_2 = 24\Big[\, & c_0^2(-2a_1^2b_1+4a_0a_1b_2)a_1^6+c_0c_1(a_1^2b_0+3a_0a_1b_1-7a_0^2b_2)a_1^6\\
&+(c_1^2+2c_0c_2)(-a_1^2b_0-a_0a_1b_1+3a_0^2b_2)a_0a_1^5\\
&+(c_1c_2+c_0c_3)(3a_1^2b_0+a_0a_1b_1-5a_0^2b_2)a_0^2a_1^4\\
&+(c_2^2+2c_1c_3+2c_0c_4)(-2a_1^2b_0+2a_0^2b_2)a_0^3a_1^3\\
&+(c_2c_3+c_1c_4)(5a_1^2b_0-a_0a_1b_1-3a_0^2b_2)a_0^4a_1^2\\
&+(c_3^2+2c_2c_4)(-3a_1^2b_0+a_0a_1b_1+a_0^2b_2)a_0^5a_1\\
&+c_3c_4(7a_1^2b_0-3a_0a_1b_1-a_0^2b_2)a_0^6+c_4^2(-4a_0a_1b_0+2a_0^2b_1)a_0^6\Big]
\end{aligned}\tag{A.22}$$

Finally the last group of 46 terms has the *type* "cb^4a^6"

$$\begin{aligned}
D_3 =\ & c_0\Big(2a_1^6b_0b_1^3-2a_0a_1^5b_1^4-4a_1^6b_0^2b_1b_2-4a_0a_1^5b_0b_1^2b_2+10a_0^2a_1^4b_1^3b_2\\
&\quad+8a_0a_1^5b_0^2b_2^2-20a_0^3a_1^3b_1^2b_2^2+20a_0^4a_1^2b_1b_2^3-8a_0^5a_1b_2^4\Big)\\
&+c_1\Big(-2a_1^6b_0^2b_1^2+2a_0a_1^5b_0b_1^3+2a_1^6b_0^3b_2+6a_0a_1^5b_0^2b_1b_2-10a_0^2a_1^4b_0b_1^2b_2\\
&\quad-10a_0^2a_1^4b_0^2b_2^2+20a_0^3a_1^3b_0b_1b_2^2-10a_0^4a_1^2b_0b_2^3-2a_0^5a_1b_1b_2^3+2a_0^6b_2^4\Big)\\
&+c_2\Big(2a_1^6b_0^3b_1-2a_0a_1^5b_0^2b_1^2-8a_0a_1^5b_0^3b_2+10a_0^2a_1^4b_0^2b_1b_2-10a_0^4a_1^2b_0b_1b_2^2\\
&\quad+2a_0^5a_1b_1^2b_2^2+8a_0^5a_1b_0b_2^3-2a_0^6b_1b_2^3\Big)
\end{aligned}$$

$$
\begin{aligned}
&+c_3\Big(-2a_1^6b_0^4+2a_0a_1^5b_0^3b1+10a_0^2a_1^4b_0^3b_2-20a_0^3a_1^3b_0^2b_1b_2+10a_0^4a_1^2b_0b_1^2b_2\\
&\qquad -2a_0^5a_1b_1^3b_2+10a_0^4a_1^2b_0^2b_2^2-6a_0^5a_1b_0b_1b_2^2+2a_0^6b_1^2b_2^2-2a_0^6b_0b_2^3\Big)\\
&+c_4\Big(8a_0a_1^5b_0^4-20a_0^2a_1^4b_0^3b_1+20a_0^3a_1^3b_0^2b_1^2-10a_0^4a_1^2b_0b_1^3+2a_0^5a_1b_1^4\\
&\qquad +4a_0^5a_1b_0b_1^2b_2-2a_0^6b_1^3b_2-8a_0^5a_1b_0^2b_2^2+4a_0^6b_0b_1b_2^2\Big) \qquad \text{(A.23)}
\end{aligned}
$$

From D_1 and D_2 one can split off a factor $\mathcal{D}_3^2$ and $24\mathcal{D}_5$, respectively, and gets (for D_2, say) a complicated polynomial (with the coefficients c_i of C_4 replaced by the coefficients of G_4 for D_1)

$$
\begin{aligned}
P &= a_1^4(-2b_1c_0+b_0c_1)-a_1^3a_0(-4b_2c_0-b_1c_1+2b_0c_2)+a_1^2a_0^2(-3b_2c_1+3b_0c_3)\\
&\quad -a_1a_0^3(-2b_2c_2+b_1c_3+4b_0c_4)+a_0^4(-b_2c_3+2b_1c_4) \qquad \text{(A.24)}
\end{aligned}
$$

The structural meaning of this expression is revealed by the following identity

$$
\det\begin{pmatrix} c_4 & c_3 & c_2 & c_1 & c_0\\ b_2^2 & 2b_2b_1 & 2b_2b_0+b_1^2 & 2b_1b_0 & b_0^2\\ a_1^2 & 2a_1a_0 & a_0^2 & 0 & 0\\ 0 & a_1^2 & 2a_1a_0 & a_0^2 & 0\\ 0 & 0 & a_1^2 & 2a_1a_0 & a_0^2\end{pmatrix} = P\cdot\det\begin{pmatrix} b_2 & b_1 & b_0\\ a_1 & a_0 & 0\\ 0 & a_1 & a_0\end{pmatrix} \qquad \text{(A.25)}
$$

(cf. section A.1). Similarly D_3 is 2 times the product of two determinants, cf. (A.10).

B Rational Curves P in X

Which (smooth) rational curves P, suitable as support for the world-sheet instanton, exist in X? As we want to bring to bear the elliptically fibered structure of X and the spectral nature of V we concentrate on horizontal curves: if $p\sigma+a_PF$ denotes the cohomology class then P is said to lie 'horizontally' (embedded in B via σ) if $a_P=0$. We first search for such *rational* base curves; then we treat the question of isolatedness.

B.1 The different rational base surfaces B

The Calabi-Yau threefold X has the following possible rational surfaces as bases B: a Hirzebruch surface $\mathbf{F_k}$ with $k=0,1,2$ (or blow-ups of it); or B is $\mathbf{P^2}$ or blow-ups of it, i.e. one of the del Pezzo surfaces dP_k with $k\le 8$; finally the Enriques surface is possible.

The surface $\mathbf{F_k}$ is a $\mathbf{P^1}$-fibration over a base $\mathbf{P_1}$ denoted by b (the fibre is denoted by f; if no confusion arises b and f will denote also the cohomology classes). One finds $c_1(\mathbf{F_k})=2b+(2+k)f$. b of $b^2=-k$ is a section of the fibration; there is another section ("at infinity") of class $b_\infty=b+kf$ of self-intersection $+k$. The Kaehler cone (the very ample classes) equals the positive (ample) classes and is given (cf. footnote 22) by the numerically effective classes $xb+yf$ with $x>0, y>kx$. An irreducible non-singular curve exists in a class $xb+yf$ exactly if the class lies in the mentioned cone or is one of the elements b, f or ab_∞ (with $a>0$) on the boundary of the cone; these classes together with their positive linear combinations span the effective cone ($x, y \geq 0$). c_1 is positive for $\mathbf{F_0}$ and $\mathbf{F_1}$, for $\mathbf{F_2}$ it lies on the boundary of the positive cone.

We will concentrate on the case $B = \mathbf{F_k}$ for illustration; in $\mathbf{P^2}$ one has (non-isolated) rational curves given by a line or a quadric (of classes l and $2l$); on a dP_k one has among various further curves the exceptional $\mathbf{P^1}$'s (from the blow-up) of self-intersection -1.

B.2 Horizontal rational curves

For B a Hirzebruch surfaces $\mathbf{F_k}$ ($k = 0, 1, 2$) let us find the possible rational instanton curves on B besides the three immediate candidates $P = b, b_\infty$ (of class $b + kf$) and f.

The cohomology class $P = xb + yf$ is represented by an irreducible smooth curve for[22] $P = b, f, a\, b_\infty$ (with $a > 0$ and $k > 0$) or $P > 0$, i.e. P ample (positive) which comes down to $P \cdot f = x > 0$ and $P \cdot b = y - kx > 0$. So, except for f and b, one has $x > 0, y > 0$ which we will assume from now on. Now the numerical rationality condition

$$2 \overset{!}{=} c_1 \cdot P - P^2 = 2(x + y) - kx + kx^2 - 2xy \tag{B.1}$$

leads to the following possibilities:
$\mathbf{F_0}$: $P = b + yf$ or $P = xb + f$
$\mathbf{F_1}$: $P = b + yf$ or $P = 2(y - 1)b + yf$
$\mathbf{F_2}$: $P = b + yf$ or $P = (y - 1)b + yf$.
Combining this finding with the requirement that P (if not equal to b or f) is either of the form $a(b + kf)$ or $P > 0$ the following possibilities remain in total

$$P = b\ , \quad f\ , \quad b + yf \quad (y \geq k) \tag{B.2}$$

together with the mirrored case $xb + f$ on $\mathbf{F_0}$ and the exceptional $P = 2b + 2f$ on $\mathbf{F_1}$.

B.3 The question of isolatedness

Now let us study which of the rational curves found so far are furthermore isolated. To make the discussion transparent we recall first the relevant facts in general (cf. also [5]). For this we decompose the problem in steps: we first study in the next two subsections the rigidity question for a base curve P with respect to the two surfaces in which P is contained, that is for B and $\mathcal{E} = \pi^{-1}(P)$. Finally we point to the decomposed nature of the problem. The upshot is that the base curve b in $\mathbf{F_1}$ remains (as the corresponding other rational blow-up curves of self-intersection (-1) in a dP_k base).

B.3.1 The deformations of P in the rational base surface B

We have a 'local' information $\mathrm{def}_B^{\mathrm{loc}}(P) := h^0(P, N_B P)$ about deformations of P in B as well as a 'global' one $\mathrm{def}_B^{\mathrm{glob}}(P) := \mathrm{def}_B(P) := h^0(B, \mathcal{O}(P)) - 1$. Using Riemann-Roch

$$\sum_{i=0}^{2} (-1)^i h^i(B, \mathcal{O}_B(P)) = h^0(P, N_B P) - h^1(P, N_B P) + 1 \tag{B.3}$$

[22] Cf. R. Hartshorne, *Algebraic Geometry*, Springer Verlag (1977).

(with $\chi(B,\mathcal{O})=1$ from Noether's theorem for our rational B) we get

$$\mathrm{def}_B(P) = h^0(P,N_BP) - h^1(P,N_BP) + s - h^2(B,\mathcal{O}_B(P)) \tag{B.4}$$

(with the *superabundance* $s = h^1(B,\mathcal{O}_B(P))$) with the local terms

$$h^0(P,N_BP) - h^1(P,N_BP) = \frac{1}{2}\chi(P) + \mathrm{deg}N_BP = \frac{Pc_1+P^2}{2} \tag{B.5}$$

Let us investigate the two higher cohomological corrections s and $h^2(B,\mathcal{O}_B(P))$ in (B.4). For any curve P on a rational surface B (like a Hirzebruch surface $\mathbf{F_n}$ or a del Pezzo surface dP_k) one has $h^2(B,\mathcal{O}_B(P))=0$ which can be seen from the exact sequence $0\to\mathcal{O}_B\to\mathcal{O}_B(P)\to\mathcal{O}_P(P)\to 0$ with associated long exact sequence (where B being rational one has $p_g(B)=h^2(B,\mathcal{O}_B)=0$ and $q(B)=h^1(B,\mathcal{O}_B)=0$)

$$\begin{aligned}0 &\to h^0(B,\mathcal{O}_B)\to h^0(B,\mathcal{O}_B(P))\to h^0(P,\mathcal{O}_P(P))\\ &\to h^1(B,\mathcal{O}_B)\to h^1(B,\mathcal{O}_B(P))\to h^1(P,\mathcal{O}_P(P))\\ &\to h^2(B,\mathcal{O}_B)\to h^2(B,\mathcal{O}_B(P))\to 0\end{aligned} \tag{B.6}$$

so $h^2(B,\mathcal{O}_B(P))=0$ and $h^1(B,\mathcal{O}_B(P))=h^1(P,\mathcal{O}_P(P))$ in (B.4), giving

$$B \text{ rational} \Longrightarrow \mathrm{def}_B^{\mathrm{glob}}(P) = \mathrm{def}_B^{\mathrm{loc}}(P) \tag{B.7}$$

The mentioned middle terms in (B.4) not only cancel but vanish individually if $h^0(P,K_P-N_BP)=0$ which happens if $0>\deg(K_P-N_BP)=P(K_B+P)-P^2=PK_B$ which is guaranteed if $-K_B$ is ample, as for $B=\mathbf{F_n}$ with[23] $n=0,1$ and dP_k $(k\neq 9)$; in general

$$P\cdot c_1>0 \Longrightarrow \mathrm{def}_B(P) = h^0(P,N_BP) = \frac{Pc_1+P^2}{2} \tag{B.8}$$

Isolated rational curves in $\mathbf{F_k}$. We restrict ourselves to *isolated* instantons; so for $p=b$, say, we restrict us to $\mathbf{F_1},\mathbf{F_2}$ where $b^2<0$ enforces isolatedness[24] in $\mathbf{F_k}$.[25] Let us complete the discussion with the case $P=b+yf$ where $y>k$: then the number of deformations of the very ample P is

$$def_{\mathbf{F_k}}(P) = 2y-k+1 \tag{B.9}$$

(here was $y>k$ but $P=b_\infty$ of $y=k$ is also covered: $h^0(b_\infty,N_{\mathbf{F_k}}b_\infty)=h^0\big(b_\infty,\mathcal{O}_{b_\infty}(k)\big)=k+1$). For f obviously $\mathrm{def}_{F_n}f=1$ (also from (B.8)). Finally (cf. (B.2)) $P=2b+2f$ on $\mathbf{F_1}$ has $def_B(P)=5$ by (B.8). Thus only b in $\mathbf{F_1}$ remains (similarly the blow-up curves of self-intersection (-1) in dP_k).

[23]For $\mathbf{F_2}$ of $c_1=2b_\infty$ the Kodaira vanishing theorem gives $s=h^1(B,\mathcal{O}(P))=h^1(B,\mathcal{O}(K-P))=0$ if $-K+P=(x+2)b+(y+4)f$ is ample, i.e. for $y>2x, x>-2$, so clearly for all ample $P=xb+yf$ (where even $x>0$) and for f; finally $kb_+\cdot c_1(\mathbf{F_2})=4k>0$ making (B.8) again applicable.

[24]Having negative self-intersection b cannot move; more formally $\mathrm{def}_B(b)=0$ on $\mathbf{F_1}$ by (B.8), and $\mathrm{def}_B(P)=\frac{Pc_1+P^2}{2}+h^1(P,N_BP)=-1+h^0(P,K_P-N_BP)=0$ on $\mathbf{F_2}$ from (B.5) and $N_Bb=K_b$.

[25]The same follows in X for $k\neq 2$ (for $k=2$ one has $\mathcal{E}_b=b\times F$ showing a deformation), cf. section B.3.2.

B.3.2 The deformations of P in the vertical elliptic surface $\mathcal{E}$

The Kodaira formula identifies the canonical bundle of $\mathcal{E}$ as a pull-back class $K_{\mathcal{E}} = \pi_{\mathcal{E}}^* K_P + \chi(\mathcal{E}, \mathcal{O}_{\mathcal{E}})F$. So $\chi(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) = \frac{1}{12}e(\mathcal{E})$. But $e(\mathcal{E}) = 12c_1 \cdot P$ as the elliptic fibration has discriminant $12c_1$ so that $\chi(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) = c_1 \cdot P$. Alternatively one can see from adjunction

$$c(\mathcal{E}) = c(P)\frac{(1+r)(1+r+2c_1)(1+r+3c_1)}{1+3r+6c_1} \tag{B.10}$$

that

$$c_1(\mathcal{E}) = \Big(e(P) - c_1 \cdot P\Big)F\,, \qquad e(\mathcal{E}) = 12c_1 \cdot P \tag{B.11}$$

Note that the number $c_1 \cdot P$ has the following important interpretation: as $c_1(\mathcal{E})$ is a pull-back class, i.e. a number of fibers, one has from (B.11) that $P_{\mathcal{E}} \cdot c_1(\mathcal{E}) = e_P - c_1 \cdot P_B$ (we write $P_{\mathcal{E}}$ when we wish to emphasize that P is considered as a curve in $\mathcal{E}$); so with adjunction $-e_P = P_{\mathcal{E}}^2 - P_{\mathcal{E}} \cdot c_1(\mathcal{E})$ inside $\mathcal{E}$ one gets that the self-intersection of $P_{\mathcal{E}}$ in $\mathcal{E}$ is $P_{\mathcal{E}}^2 = -c_1 \cdot P_B = -\chi(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$. So with $\deg(N_{\mathcal{E}}\, P_{\mathcal{E}}) = P_{\mathcal{E}}^2$ one has the criterion

$$P \cdot c_1 > 0 \Longrightarrow def_{\mathcal{E}}^{\text{loc}}\, P_{\mathcal{E}} = 0 \tag{B.12}$$

So, assuming $P \cdot c_1 > 0$ as in (B.8), we have no further deformations in the vertical direction except for the case $P = b$ of $b \cdot c_1 = 0$ in $B = \mathbf{F_2}$: there $\mathcal{E} = b \times F$ shows a deformation in X (while one has no deformation in the base B, cf. section B.3.1).

Examples. The cases $c_1 \cdot P = 0, 1, 2$ lead to $\mathcal{E} = b \times F, dP_9, K3$ of $e(\mathcal{E}) = 0, 12, 24$. These $\mathcal{E}$ occur over $P = b$ in $B = \mathbf{F_k}$ with $k = 2, 1, 0$ (cf. section 3.1): first, the ruled surface $b \times F$ over F, being one of the two exceptional divisors (the other is the base $\mathbf{F_2}$ itself) of the STU Calabi-Yau $\mathbf{P}_{1,1,2,8,12}(24)$; secondly, the rational elliptic dP_9 surface which occurs also over any exceptional curve ($b^2 = -1, b \cdot c_1 = 1$; rationality and the second property imply the first) in a dP_k base; finally for $\mathbf{F_0}$ one gets the well-known $K3$ fibers, which occur also over each fiber of a $\mathbf{F_k}$ base ($c_1 \cdot P = 2$ by adjunction).

B.3.3 The deformations of P in X

The total deformation space can be considered fibered together out of the pieces investigated so far. Concerning $def_X^{\text{loc}}\, P$ consider

$$0 \to N_B\, P \to N_X\, P \to N_X\, B|_P \to 0 \tag{B.13}$$

the last term being $N_{\mathcal{E}}\, P$. To get $def_X^{\text{loc}}\, P = def_B^{\text{loc}}\, P$ one has to show that $H^0(P, N_X\, B|_P) = 0$, i.e. that there are no further deformations of P in $\mathcal{E}$. This will hold if $\deg N_{\mathcal{E}}\, P = P_{\mathcal{E}}^2 = -\chi < 0$, cf. (B.12).

C Some Lemmata

C.1 Lemma 1

We follow the notation in section 4 where one finds that b contributes iff $V|_b$ is trivial

$$W_b \neq 0 \Longleftrightarrow V|_b = \bigoplus_{i=1}^{n} \mathcal{O}_b \tag{C.1}$$

A corresponding framing would give n linearly independent global sections such that

$$W_b \neq 0 \Longrightarrow h^0(b, V|_b) = n \tag{C.2}$$

This is also directly a consequence of $W_b \neq 0 \Longleftrightarrow h^0\Big(b, V|_b(-1)\Big) = 0$ in (4.1). For this note that the short exact sequence

$$0 \longrightarrow l(-F)|_c \longrightarrow l|_c \longrightarrow \mathbf{C^n} \longrightarrow 0 \tag{C.3}$$

and its associated long exact cohomology sequence

$$0 \longrightarrow H^0(c, l(-F)|_c) \longrightarrow H^0(c, l|_c) \longrightarrow \mathbf{C^n} \longrightarrow H^1(c, l(-F)|_c) \longrightarrow H^1(c, l|_c) \longrightarrow 0 \tag{C.4}$$

show that $h^0\Big(b, V|_b(-1)\Big) = h^0\Big(c, l(-F)|_c\Big) = 0 \Longrightarrow h^0(b, V|_b) = h^0(c, l|_c) = n$ as one has

$$h^0\Big(c, l(-F)|_c\Big) = h^1\Big(c, l(-F)|_c\Big) \tag{C.5}$$

This follows either, arguing downstairs on b, using $h^i\Big(c, l(-F)|_c\Big) = h^i\Big(b, V|_b(-1)\Big)$ from

$$h^0\Big(b, V|_b(-1)\Big) - h^1\Big(b, V|_b(-1)\Big) = \int_b c_1\Big(V|_b(-1)\Big) + \frac{c_1(b)}{2} = \int_b c_1(V)|_b = 0 \tag{C.6}$$

or, with (3.26), also directly upstairs on c as one has with (3.26)

$$h^0\Big(c, l(-F)|_c\Big) - h^1\Big(c, l(-F)|_c\Big) = \frac{1}{2}\deg K_c + \frac{1}{2}\deg K_c^{-1} = 0 \tag{C.7}$$

C.2 Lemma 2

As $\pi_{\mathcal{E}}$ is a projection one has $\pi_{\mathcal{E}}^{-1}(b) = \mathcal{E}$ and so $H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\alpha s+\beta F)\Big) = H^0\Big(b, \pi_{\mathcal{E}*}\mathcal{O}_{\mathcal{E}}(\alpha s+\beta F)\Big)$. From the Leray spectral sequence for the elliptically fibered surface $\mathcal{E}$ one has

$$\begin{aligned} 0 &\longrightarrow H^1\Big(b, \pi_*\mathcal{O}_{\mathcal{E}}(\alpha s)\Big) \longrightarrow H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\alpha s)\Big) \longrightarrow H^0\Big(b, R^1\pi_*\mathcal{O}_{\mathcal{E}}(\alpha s)\Big) \\ &\longrightarrow 0 \longrightarrow H^2\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\alpha s)\Big) \longrightarrow H^1\Big(b, R^1\pi_*\mathcal{O}_{\mathcal{E}}(\alpha s)\Big) \longrightarrow 0 \end{aligned} \tag{C.8}$$

The case $\alpha > 0$. For $\alpha > 0$ one has (recall that $\mathcal{L}|_b = K_B^{-1}|_b = \mathcal{O}_b(\chi)$)

$$\pi_*\mathcal{O}_{\mathcal{E}}(\alpha s + \beta F) = \mathcal{O}_b(\beta) \oplus \bigoplus_{i=2}^{\alpha} \mathcal{O}_b\Big(\beta - i\chi\Big) \tag{C.9}$$

$$R^1\pi_*\mathcal{O}_{\mathcal{E}}(\alpha s + \beta F) = 0 \tag{C.10}$$

and the Leray spectral sequence gives $H^2\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)\Big) = 0$ and

$$H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)\Big) = H^1\Big(b, \pi_{\mathcal{E}*}\mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)\Big) \tag{C.11}$$

One finds that $h^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)\Big)$ vanishes, if $\alpha > 0$, just for negative β

$$\alpha > 0: \qquad h^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)\Big) = h^0\Big(b, \pi_{\mathcal{E}*}\mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)\Big) = 0 \iff \beta < 0 \tag{C.12}$$

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More precisely note that (where $\{m\} := m+1 = h^0(b, \mathcal{O}_b(m))$ for $m \geq 0$ or else zero)

$$h^0\Big(\mathcal{E}, \mathcal{O}_\mathcal{E}(\alpha s + \beta F)\Big) = \{\beta\} + \sum_{i=2}^{\alpha} \{\beta - i\chi\} \to \alpha(\beta+1) - \left(\frac{\alpha(\alpha+1)}{2} - 1\right)\chi \tag{C.13}$$

$$h^1\Big(\mathcal{E}, \mathcal{O}_\mathcal{E}(\alpha s + \beta F)\Big) = \{-\beta-2\} + \sum_{i=2}^{\alpha} \{-\beta - 2 + i\chi\} \to -\alpha(\beta+1) + \left(\frac{\alpha(\alpha+1)}{2} - 1\right)\chi \tag{C.14}$$

Here the final evaluations for β or $-\beta$ sufficiently big, i.e. $\beta - \alpha\chi \geq 0$ or $-\beta - 2 \geq 0$; actually (C.14) still holds for $\beta = -1$.

The case $\alpha = 0$. Next, in the case $\alpha = 0$ one has $\pi_* \mathcal{O}_\mathcal{E} = \mathcal{O}_b$ and $R^1\pi_*\mathcal{O}_\mathcal{E} = K_B|_b = \mathcal{O}_b(-\chi)$. One finds

$$\begin{aligned} H^1(b, R^1\pi_*\mathcal{O}_\mathcal{E}(0s + \beta F)) &\cong H^1(b, \mathcal{O}_b(\beta - \chi)) \cong H^0(b, \mathcal{O}_b(\chi - \beta - 2))^* && \text{(C.15)}\\ H^0(b, R^1\pi_*\mathcal{O}_\mathcal{E}(0s + \beta F)) &\cong H^0(b, \mathcal{O}_b(\beta - \chi)) && \text{(C.16)} \end{aligned}$$

The case $\alpha < 0$. Finally in the case $\alpha < 0$ one has

$$\begin{aligned} \pi_*\mathcal{O}_\mathcal{E}(\alpha s) &= 0 && \text{(C.17)}\\ \Big(R^1\pi_*\mathcal{O}_\mathcal{E}(\alpha s)\Big)^* &= \pi_*\Big(\mathcal{O}_\mathcal{E}(-\alpha s) \otimes (K_\mathcal{E} \otimes K_b^{-1})\Big) = \pi_*\mathcal{O}_\mathcal{E}(-\alpha s) \otimes K_B^{-1}|_b && \text{(C.18)} \end{aligned}$$

and finds

$$\begin{aligned} H^1\left(b, R^1\pi_*\mathcal{O}_\mathcal{E}(\alpha s + \beta F)\right) &= H^0\left(b, \left(\mathcal{O}_b \oplus \bigoplus_{i=2}^{-\alpha} \mathcal{O}_b(-i\chi)\right) \otimes \mathcal{O}_b(\chi - \beta - 2)\right)^* && \text{(C.19)}\\ H^0\left(b, R^1\pi_*\mathcal{O}_\mathcal{E}(\alpha s + \beta F)\right) &= H^0\left(b, \left(\mathcal{O}_b \oplus \bigoplus_{i=2}^{-\alpha} \mathcal{O}_b(i\chi)\right) \otimes \mathcal{O}_b(\beta - \chi)\right) && \text{(C.20)} \end{aligned}$$

To summarize: we get that $H^1\big(b, R^1\pi_*\mathcal{O}_\mathcal{E}(\alpha s + \beta F)\big)$ vanishes[26] and that $H^0(b, R^1\pi_*\mathcal{O}_\mathcal{E}(\alpha s + \beta F))$ is vanishing[27] in the following cases

$$H^2(\mathcal{E}, \mathcal{O}_\mathcal{E}(\alpha s + \beta F)) = 0 \Longleftarrow \begin{cases} \beta \text{ arbitrary} & \text{for } \alpha > 0 \\ \beta > \chi - 2 & \text{for } \alpha \leq 0 \end{cases} \tag{C.21}$$

$$H^1(\mathcal{E}, \mathcal{O}_\mathcal{E}(\alpha s + \beta F)) \cong H^1(b, \pi_*\mathcal{O}_\mathcal{E}(\alpha s + \beta F)) \Longleftarrow \begin{cases} \beta \text{ arbitrary} & \text{for } \alpha > 0 \\ \beta < (\alpha+1)\chi & \text{for } \alpha \leq 0 \end{cases} \tag{C.22}$$

Example. For $\lambda = 1/2$ one has also to consider the case $p = n(\lambda - \frac{1}{2}) = \alpha - n = 0$ by (4.30), so $\chi - 2 < q = \beta - r < \chi$ from the conditions (C.21), (C.22) for $\mathcal{O}_\mathcal{E}(ps + qF)$; but the resulting consequence $\beta - r = -r + \frac{1+n}{2}\chi - 1 \geq \chi - 1$ (from the lower bound), i.e. $r - \frac{n-1}{2}\chi - 1 < 0$ contradicts (3.11) or (3.15); so then $H^2(\mathcal{E}, \mathcal{O}_\mathcal{E}(\alpha s + \beta F)) \neq 0$.

[26] Such that then one will have, from (C.8), still $H^2(\mathcal{E}, \mathcal{O}_\mathcal{E}(\alpha s)) = 0$, leaving in the long exact sequence (4.11) again only the three H^0- and the three H^1-terms

[27] such that (C.11) holds.

C.3 Lemma 3 (cf. section 4.3, eq. (4.26))

A sufficient criterion for the condition in (4.12) to hold (necessary for $W_b \neq 0$) is $\beta < 0$

$$h^0\Big(\mathcal{E},\, l(-F-c)\Big) \; = h^0\Big(\mathcal{E},\, l(-F)\Big) \quad \rightleftarrows \quad \beta < 0 \tag{C.23}$$

as then both dimensions vanish by (4.23). The converse holds also: the dimensions can be equal only if both vanish. So assume the lhs of (C.23) and $\beta \geq 0$ such that

$$\begin{aligned} h^0\Big(\mathcal{E}, l(-F)\Big) \; &= \beta + 1 + \sum_{i=2}^{\alpha} \{\beta - i\chi\} \\ h^0\Big(\mathcal{E}, l(-F-c)\Big) \; &= \{\beta - r\} + \sum_{i=2}^{\alpha-n} \{\beta - r - i\chi\}\,, \end{aligned} \tag{C.24}$$

from $l(-F) = \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)$ with $\alpha := n(\lambda + \frac{1}{2})$. Then one gets

$$\sum_{i=2}^{\alpha-n} \Big(\{\beta - i\chi\} - \{\beta - r - i\chi\}\Big) + \sum_{i=\alpha-n+1}^{\alpha} \{\beta - i\chi\} = \{\beta - r\} - (\beta + 1) \tag{C.25}$$

(note that $\lambda > 1/2$ by our standing technical assumption (4.24), so $\alpha > n$). This leads to a contradiction: the lhs of (C.25) is always ≥ 0 as $r \geq 0$ by (3.11); so $\beta \geq r$ (otherwise the rhs would be < 0) and the rhs is $-r \leq 0$, such that $r = 0$. But then (3.11) gives $\chi = 0$ (i.e. $B = \mathbf{F_2}$) and therefore from (4.21) the contradiction $\beta = -1$. Therefore actually $\beta < 0$ and both H^0-terms on the lhs of (C.23) are zero by (C.12).

So, indeed, the necessary criterion (for $W_b \neq 0$) that the equality on the lhs of (C.23) holds is equivalent to β being negative, which also means that the two H^0-dimensions were actually zero.

D An alternative method

For another method to get a 4-term exact sequence with $H^0(c, l(-F)|_c)$ as kernel of a map ρ between spaces of equal dimension (cf. after (4.16)) choose, besides the varying curve $c \subset \mathcal{E}$, a second fixed effective divisor $c' \subset \mathcal{E}$ which may be reducible (some fibers, say) or even non-reduced (some fibers coalescing); assume $\deg D' > \deg K_c^{1/2}$ where $D' := c'|_c$. From $\deg \mathcal{L} = \deg K_c^{1/2}$ with $\mathcal{L} = l(-F) = \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)$ one has $(\deg(K_c - \mathcal{L}|_c(D')) < 0)$

$$0 \longrightarrow H^0(c, \mathcal{L}|_c) \longrightarrow H^0(c, \mathcal{L}|_c(D')) \stackrel{\rho}{\longrightarrow} H^0(D', \mathcal{L}|_c(D')|_{D'}) \longrightarrow H^1(c, \mathcal{L}|_c) \longrightarrow 0 \tag{D.1}$$

(middle terms have equal dimension: $h^0(c, \mathcal{L}|_c(D')) = \deg D' = c' \cdot c = h^0(D', \mathcal{L}|_c(D')|_{D'})$).

Let c' be a set of $m \geq r - \frac{n-1}{2}\chi$ fibers. For $m > -\beta + r - 2 + (\alpha - 3)\chi$ one has

$$0 \longrightarrow H^0(\mathcal{E}, \mathcal{L}(mF - c)) \longrightarrow H^0(\mathcal{E}, \mathcal{L}(mF)) \longrightarrow H^0(c, \mathcal{L}(mF)|_c) \longrightarrow 0 \tag{D.2}$$

Similarly here also the third term in (D.1) can be represented as

$$0 \longrightarrow \oplus_{i=1}^m H^0(F, \mathcal{O}_{F_i}(\alpha - n)) \longrightarrow \oplus_{i=1}^m H^0(F, \mathcal{O}_{F_i}(\alpha)) \longrightarrow \oplus_{i=1}^m H^0(F_i|_c, \mathcal{O}_{F_i}(\alpha)|_c) \longrightarrow 0$$

All of these sequences are interwoven with our original sequence after (4.16)

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & \rightarrow & H^0(c,\mathcal{L}|_c) & \stackrel{\delta}{\rightarrow} & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \rightarrow & & H^0(\mathcal{E},\mathcal{L}(mF-c)) & \rightarrow & H^0(\mathcal{E},\mathcal{L}(mF)) & \rightarrow & H^0(c,\mathcal{L}(mF)|_c) & \rightarrow 0 & \\
 & & \downarrow & & \downarrow r_m & & \downarrow \rho_m & & \\
0 \rightarrow & & \bigoplus_{i=1}^{m} H^0(F_i,\mathcal{O}_{F_i}(\alpha-n)) & \rightarrow & \bigoplus_{i=1}^{m} H^0(F_i,\mathcal{O}_{F_i}(\alpha)) & \rightarrow & \bigoplus_{i=1}^{m} H^0(F_i|_c,\mathcal{O}_{F_i}(\alpha)|_c) & \rightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\stackrel{\delta}{\rightarrow} & & H^1(\mathcal{E},\mathcal{L}(-c)) & \rightarrow & H^1(\mathcal{E},\mathcal{L}) & \rightarrow & H^1(c,\mathcal{L}|_c) & \rightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here one gets again[28] that $Pfaff(t) = 0 \Longleftrightarrow \det \rho_{c'}(t) = 0$. As an example take $m \geq 3$ in example 1: among the $\mathcal{O}(9s + yF)$ with $y \geq 2$ those of $y \geq 9$ have all global sections arising from (restriction from) $\mathcal{E}$, cf. (D.2).

E The cases $\lambda < -1/2$

To treat also $\lambda < -\frac{1}{2}$, outside the range (4.24), let $\widetilde{\lambda} := -\lambda > \frac{1}{2}$ and note that with

$$
l(-F) \ = \mathcal{O}_{\mathcal{E}}\left(n\left(\lambda + \frac{1}{2}\right)s + \beta F\right) \tag{E.1}
$$

$$
\widetilde{l}(-F) := \mathcal{O}_{\mathcal{E}}\left(n\left(\widetilde{\lambda} + \frac{1}{2}\right)s + \widetilde{\beta} F\right) \tag{E.2}
$$

(where $\beta = \beta(\lambda), \widetilde{\beta} = \beta(\widetilde{\lambda})$) one gets with (C.5) and (3.26) that

$$
\begin{aligned}
h^0\left(c, l(-F)|_c\right) &= h^1\left(c, l(-F)|_c\right) = h^0\left(c, \left(K_c^{1/2}\otimes \mathcal{F}_\lambda\right)^* \otimes K_c\right) = h^0\left(c, K_c^{1/2}\otimes \mathcal{F}_{-\lambda}\right) \\
&= h^0\left(c, \widetilde{l}(-F)|_c\right)
\end{aligned} \tag{E.3}
$$

So, by (4.9) and (E.3), the question of contribution (or not) is independent of the sign of λ.

So, following (4.9), one can work equally well with the new h^0-expression for $\widetilde{l}$, i.e., when one wants to consider (E.1) with $\lambda < -\frac{1}{2}$ one applies the same arguments as before to (E.2) assuming the necessary condition (4.12), i.e. $\beta < 0$, what leads again to a map (4.30), now for $\widetilde{l}$ with $\widetilde{\lambda} > \frac{1}{2}$. The map $H^1(\mathcal{E}, l(-F-c)) \longrightarrow H^1(\mathcal{E}, l(-F))$ becomes with Serre duality $H^1(\mathcal{E}, l(-F-c)^* \otimes K_{\mathcal{E}})^* \longrightarrow H^1(\mathcal{E}, l(-F)^* \otimes K_{\mathcal{E}})^*$ which is dual to

$$
H^1(\mathcal{E}, l(-F)^* \otimes K_{\mathcal{E}}) \longrightarrow H^1(\mathcal{E}, l(-F-c)^* \otimes K_{\mathcal{E}}) \tag{E.4}
$$

[28] The map ρ_m over c comes from the moduli-independent restriction map r_m. The moduli-dependence can be understood from the necessity to select representatives inside $H^0(\mathcal{E}, \mathcal{L}(mF))$ of a set of basis elements in $H^0(c, \mathcal{L}(mF)|_c)$; for this one has to take into account the equivalences arising from embedding $H^0(\mathcal{E}, \mathcal{L}(mF-c))$ into $H^0(\mathcal{E}, \mathcal{L}(mF))$ via multiplication with the defining polynomial w_c of c.

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This amounts to the original map (4.30), now for $\widetilde{l}$ which has $\widetilde{\lambda} > 1/2$,

$$H^1(\mathcal{E}, \widetilde{l}(-F-c)) \longrightarrow H^1(\mathcal{E}, \widetilde{l}(-F)) \tag{E.5}$$

as $l(-F)^* \otimes \mathcal{O}_{\mathcal{E}}(c) \otimes K_{\mathcal{E}} = \widetilde{l}(-F)$ because $l(-F) = (K_{\mathcal{E}} \otimes \mathcal{O}_{\mathcal{E}}(c))^{1/2} \otimes \underline{\mathcal{G}}_{\lambda}|_{\mathcal{E}}$ by (3.25) and $\widetilde{l}(-F)$ arises from $l(-F)$ by going from λ to $\widetilde{\lambda} = -\lambda$ (the argument we gave in (E.3) on c). So, as necessary condition for contribution to the superpotential one has to consider $\beta(\widetilde{\lambda}) < 0$ also in the case $\lambda < -\frac{1}{2}$.

Example 1: the case with $\lambda = -5/2$

Consider $\lambda = -5/2$ where $\beta = 3r + \frac{1-5n}{2}\chi - 1$ and $l(-F) = \mathcal{O}_{\mathcal{E}}(-2ns + \beta F)$ and apply the usual reasoning to $\widetilde{l}(-F)$ in (E.2) where $\widetilde{\lambda} = +5/2$ and $\widetilde{\beta} = -\beta + r - 2 + \chi$. Assuming the necessary condition (4.12), i.e. $\widetilde{\beta} < 0$, consider the map (E.5)

$$I_r \ : \ H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}\big(2ns + (\widetilde{\beta} - r)F\big)\Big) \longrightarrow H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}\big(3ns + \widetilde{\beta}F\big)\Big) \tag{E.6}$$

Concretely, consider an SU(3) bundle over $B = \mathbf{F_1}$ and take $r = 4$ such that $\beta = 4, \widetilde{\beta} = -1$ and $l(-F) = \mathcal{O}_{\mathcal{E}}(-6s + 4F), \widetilde{l}(-F) = \mathcal{O}_{\mathcal{E}}\big(9s - F\big)$. One gets a map[29] (9.11) between 44-dimensional spaces (by (C.14)).

F Reduction cases with equality condition

F.1 The case $\bar{\alpha} > n$

Note that $\bar{\alpha} \leq \frac{1}{p}\alpha = \frac{n}{p}(\lambda + \frac{1}{2})$ gives here $\lambda > p - \frac{1}{2}$. Then $\bar{\beta} \leq \frac{1}{p}\beta$ gives with (6.5)

$$\bar{\alpha} \geq \frac{1}{p}\alpha + \frac{p-1}{p}\frac{n}{r - n\chi}\left(r - \frac{n-1}{2}\chi - 1\right) \tag{F.1}$$

such that

$$r - \frac{n-1}{2}\chi - 1 \leq 0 \tag{F.2}$$

With (3.12) one finds that $\chi = 0, r = 1$ such that one gets with (6.5) (for $\lambda + \frac{1}{2} \in p\mathbf{Z}^{>1}$)

$$l(-F) = \mathcal{O}_{\mathcal{E}}\left(n\left(\lambda + \frac{1}{2}\right)s - \left(\lambda + \frac{1}{2}\right)F\right) \tag{F.3}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\left(\frac{n(\lambda + \frac{1}{2})}{p}s - \frac{\lambda + \frac{1}{2}}{p}F\right) \tag{F.4}$$

[29]Which is dual to the map arising via Serre duality from the map $H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(-9s + 0F)) \longrightarrow H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(-6s + 4F))$.

F.2 The case $\bar{\alpha} = n$

Note that $h^0(c, \bar{l}(-F)|_c) > 0$ needs $\deg \bar{l}(-F)|_c \geq 0$; but in the case (6.18) one has $\deg \bar{l}(-F)|_c = n(\frac{n-2}{n-1}r - (\frac{n}{2} - 1)\chi - 1)$ so we do not have to consider SU(2) bundles here.

Here $\bar{\alpha} \leq \frac{1}{p}\alpha = \frac{n}{p}(\lambda + \frac{1}{2})$ gives $\lambda \geq p - \frac{1}{2}$ and the $\bar{\beta}$-bound gives with (6.8)

$$\left(\lambda - \left(\frac{1}{2} + \frac{p}{n-1}\right)\right)(r - n\chi) - \frac{1}{2}\left(\frac{2p}{n-1} - n(p-1) + 1\right)\chi \leq p - 1 \tag{F.5}$$

Thereby one derives (after checking the sign of the prefactor) with (4.28) that one has $\left[\left(\lambda - (\frac{1}{2} + \frac{p}{n-1})\right)\frac{\frac{n+1}{2}}{\lambda - \frac{1}{2}} - \frac{1}{2}\left(\frac{2p}{n-1} - n(p-1) + 1\right)\right]\chi \leq p - 1$ or

$$\left(n - 2 - \frac{1}{\lambda - \frac{1}{2}}\right)\chi \leq 2\,\frac{p-1}{p}\,\frac{n-1}{n+1} \tag{F.6}$$

For $p = 1$ this can be fulfilled by $\chi = 0$ where then (F.5) gives $\lambda \leq \frac{1}{2}\frac{n+1}{n-1}$, allowing just $\lambda = \frac{3}{2}$ for $n = 2$, or, for even r, also $\lambda = 1$; but we need $n > 2$ so let us now assume that $\chi \geq 1$; then, for $n > 2$, one can have SU(3) bundles with $\lambda = \frac{3}{2}$ where one gets $2\chi \leq r - 3\chi \leq 2\chi$ from (F.5) and (4.29) such that $r = 5\chi$ (case $\sharp$ 2 there); or SU(4) bundles with $\lambda = 1$ where one gets $5\chi \leq r - 4\chi \leq 5\chi$ such that $r = 9\chi$ (case $\sharp$ 1 there).

So let now $p \geq 2$. One derives then

$$(n-3)\chi \;\leq\; \left(n - 2 - \frac{1}{p-1}\right)\chi \leq 2\,\frac{n-1}{n+1} \tag{F.7}$$

SU(3) bundles. Here, where $\lambda \in \frac{1}{2} + \mathbf{Z}$, one gets from (F.7) that $\frac{p-2}{p-1}\chi \leq 1$; we will discuss the case $p = 2$ separately. So let $p > 2$ such that[30] $\chi = 0$ or 1; more precisely one has from (F.6)

$$\frac{\lambda - \frac{3}{2}}{\lambda - \frac{1}{2}}\chi \leq \frac{p-1}{p} \tag{F.8}$$

For $\chi = 0$, where r must be even by (6.8), one gets from (F.5) that $(\frac{p}{2} - 1)r \leq (\lambda - \frac{p+1}{2})r \leq p - 1$; so $r = 2$ or $r = 4, p = 3$; here $r = 2$ (case $\sharp$ 1) leads to $\lambda \leq p$, so $\lambda = p - \frac{1}{2}$ and

$$\text{case } \sharp 1 \qquad l(-F) = \mathcal{O}_{\mathcal{E}}\left(3\left(\lambda + \frac{1}{2}\right)s - 2\lambda F\right) \tag{F.9}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\left(3s - 2F\right) \tag{F.10}$$

whereas the $r = 4, p = 3$ case (case $\sharp$ 4) leads to $\lambda \leq \frac{5}{2}$ such that $\lambda = p - \frac{1}{2} = \frac{5}{2}$ and

$$\text{case } \sharp 4 \qquad l(-F) = \mathcal{O}_{\mathcal{E}}\left(9s - 9F\right) \tag{F.11}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\left(3s - 3F\right) \tag{F.12}$$

[30] The case $p = 3, \chi = 2$ with $\lambda = 3/2$ from (F.8) contradicts $\lambda \geq p - \frac{1}{2}$.

For $\chi = 1$ one gets $p - \frac{1}{2} \leq \lambda \leq p + \frac{1}{2}$ from (F.8), giving $r \geq 3 + \frac{2}{p-1}$ for $\lambda = p - \frac{1}{2}$ and $r \geq 3 + \frac{2}{p}$ for $\lambda = p + \frac{1}{2}$ by (4.28); here $\frac{1}{2}(r-3) = (\lambda - \frac{p+1}{2})(r-3) \leq 1$ from (F.5), leaving only the case $p = 3, \lambda = p - \frac{1}{2} = \frac{5}{2}$ and $r = 5$ (case $\sharp$ 5) as r must be odd by (6.8)

$$\text{case } \sharp 5 \qquad l(-F) = \mathcal{O}_{\mathcal{E}}\Big(9s - 3F\Big) \tag{F.13}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\Big(3s - F\Big) \tag{F.14}$$

Finally, for $p = 2$ now (F.5) says that $(\lambda - \frac{3}{2})(r - 3\chi) \leq 1$, so $\lambda = 3/2$ (case $\sharp$ 3) is a solution (the only one[31]), so one has for $r \geq 5\chi$ and $r \equiv \chi \,(\mathrm{mod}\, 2)$ (with $\deg\det \bar{\iota}_1 = \frac{3r-5\chi}{2}$)

$$\text{case } \sharp 3 \qquad l(-F) = \mathcal{O}_{\mathcal{E}}(6s - (r - 5\chi + 1)F) \tag{F.15}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\left(3s - \left(\frac{r - 5\chi}{2} + 1\right)F\right) \tag{F.16}$$

Let us also recall the earlier mentioned $p = 1$ case

$$\text{case } \sharp 2 \qquad l(-F) = \mathcal{O}_{\mathcal{E}}\Big(6s - F\Big) \tag{F.17}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\Big(3s - F\Big) \tag{F.18}$$

SU(4) bundles. Here one gets from (F.7) that $\chi = 0$ or 1. Actually[32] $\chi = 0$ and one gets, with $\lambda - \frac{1}{2} \geq p - 1$ from the remark before (F.5), that $(\frac{2}{3}p - 1)r \leq p - 1$ or $r \leq 1 + \frac{p}{2p-3}$, such that $p = 2, r = 3$ (case $\sharp$ 2) as $3|r$ by (6.8); one gets from (F.5) that $\lambda = \frac{3}{2}$ and one has (with $\deg\det \bar{\iota}_1 = 4$ and $\deg\det \iota_1 = 24$)

$$\text{case } \sharp 2 \qquad l(-F) = \mathcal{O}_{\mathcal{E}}\Big(8s - 4F\Big) \tag{F.19}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\Big(4s - 2F\Big) \tag{F.20}$$

Let us also recall the earlier mentioned $p = 1$ case

$$\text{case } \sharp 1 \qquad l(-F) = \mathcal{O}_{\mathcal{E}}\Big(6s - F\Big) \tag{F.21}$$

$$\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}\Big(4s - F\Big) \tag{F.22}$$

SU(n) bundles with $n \geq 5$. Here one gets $\chi = 0$ such that $(p - 1 - \frac{p}{n-1})r \leq (\lambda - \frac{1}{2} - \frac{p}{n-1})r \leq p - 1$ or $(1 - \frac{2}{n-1})r \leq (1 - \frac{1}{n-1}\frac{p}{p-1})r \leq 1$; this has no solutions as $(n-1)|r$ by (6.8).

So in total one gets the following lists

[31] For, if $\lambda > 3/2$, note that $\frac{2}{\lambda - \frac{1}{2}}\chi \leq r - 3\chi \leq \frac{1}{\lambda - \frac{3}{2}}$ by (4.29) such that $2\frac{\lambda - \frac{3}{2}}{\lambda - \frac{1}{2}}\chi \leq 1$ which implies $\lambda = 5/2$ (as for $\lambda \geq 7/2$ the coefficient of χ becomes > 1 which enforces $\chi = 0$ where (F.5) gives the contradiction $(\lambda - \frac{3}{2})r \leq 1$ as $r > 0$). $\lambda = 5/2$ is excluded as one gets $r - 3\chi \leq 1$ while $\chi \leq r - 3\chi$ by (4.29): $\chi = 0$ giving $r = 1$ and $\chi = 1$ giving $r = 4$ are both excluded as (6.8) must be integral.

[32] For $\chi = 1$ one gets from (F.5) that $(\frac{2}{3}p - 1)(r - 4) \leq (\lambda - \frac{1}{2} - \frac{p}{3})(r - 4) \leq -\frac{2}{3}p + \frac{3}{2}$ or $r \leq 3 + \frac{3/2}{2p-3}$, i.e. $r = 3$ as $3|r$ by (6.8) (also here $\lambda \in \mathbf{Z}$ and $p = 2$ by (F.7)); so (3.11) is violated.

- SU(3) bundles

$\sharp$	χ	r	λ	p
1	0	2	$p-\frac{1}{2}$	>2
2	≥ 1	5χ	$\frac{3}{2}$	1
3	χ	$\geq 5\chi, \equiv \chi(2)$	$\frac{3}{2}$	2
4	0	4	$\frac{5}{2}$	3
5	1	5	$\frac{5}{2}$	3

Here the case 1 for $p=2$ is the case 3 for $\chi=0, r=2$. Note that in case 3 one has also the assumption $r\geq 5\chi$.

- SU(4) bundles

$\sharp$	χ	r	λ	p
1	≥ 1	9χ	1	1
2	0	3	$\frac{3}{2}$	2

F.3 SU(3) and SU(4) bundles with $Pfaff\equiv 0$

F.3.1 SU(3) bundles

According to section 7 one uses here besides the strong reduction condition $p\bar{\beta}\leq\beta$ the condition that (7.2) is positive

$$-\frac{1}{p}\beta \;\leq\; -\bar{\beta} \;<\; \frac{r-5\chi}{2}+1 \tag{F.23}$$

Furthermore the other part $p\bar{\alpha}\leq\alpha$ of the strong reduction condition gives (with $\bar{\alpha}\geq 3$) that $p\leq\lambda+\frac{1}{2}$ such that one gets

$$2\left(\lambda-\frac{1}{2}\right)r-2\left(3\lambda+\frac{1}{2}\right)\chi+2 \;<\left(\lambda+\frac{1}{2}\right)r-5\left(\lambda+\frac{1}{2}\right)\chi+2\left(\lambda+\frac{1}{2}\right) \tag{F.24}$$

or equivalently

$$\left(\lambda-\frac{3}{2}\right)(r-\chi)<2\left(\lambda-\frac{1}{2}\right) \tag{F.25}$$

Let us first assume that $\lambda=3/2$. Then one gets $\frac{1}{p}(r-5\chi+1)\leq-\bar{\beta}<\frac{r-5\chi}{2}+1$ from (F.23); as $p\leq 2$ and $r-5\chi\geq 0$ by (4.28) one gets $p=2$ with $\frac{r-5\chi}{2}+\frac{1}{2}\leq-\bar{\beta}<\frac{r-5\chi}{2}+1$; the two ensuing cases $r\equiv\chi(2)$, where no (integral !) solution for $\bar{\beta}$ exists, and $r\not\equiv\chi(2)$ where $\bar{\beta}=-(\frac{r-5\chi+1}{2})$ are discussed further in section 7. We show now that no further cases exist.

Let us therefore assume $\lambda\neq 3/2$; as $n=3$ one has $\lambda=\frac{1}{2}+m$ with $m\in\mathbf{Z}^{\geq 1}$, so let now $m\neq 1$. From (4.28) one gets

$$r-\chi\geq 2\,\frac{m+1}{m}\chi \tag{F.26}$$

such that one gets from (F.25) that $\chi \leq \frac{m^2}{m^2-1}$ and therefore $\chi = 0$ or 1.

For $\chi = 0$ one has $r < 2\frac{m}{m-1}$ from (F.25) such that $m = 2$ with $r = 1, 2, 3$ or $m \geq 3$ with $r = 1, 2$; further $p \leq m+1$. The $Pfaff \equiv 0$ case $\chi = 0, r = 1$ was already covered in section 6.3.1. From $\frac{1}{p}(mr+1) \leq -\bar{\beta} < \frac{r}{2} + 1$ one gets for $r = 2$ that $-\bar{\beta} = 1$ (recall $\bar{\beta} < 0$) such that one gets the contradiction $2m + 1 \leq p$; for $r = 3, m = 2$ one gets the contradiction $\frac{1}{p}7 \leq -\bar{\beta} < \frac{5}{2}$ (allowing no integral $\bar{\beta}$ solution for $p = 3$) where $p \leq 3$.

For $\chi = 1$ one gets $r - 1 < 2\frac{m}{m-1}$ from (F.25) contradicting $r - 1 \geq 2\frac{m+1}{m-1}$ from (F.26).

F.3.2 SU(4) bundles

Following the same procedure one finds that here no further solutions exist besides the already covered case $\chi = 0, r = 1$. For $\lambda \in \frac{1}{2} + \mathbf{Z}$ one finds $\chi = 0$ and a contradiction for $r \neq 1$; for $\lambda \in \mathbf{Z}$ one has the same for $\lambda \neq 1$ and gets for $\lambda = 1$ a contradiction to (4.28).

G Explicit matrix representations for $n = 3, \lambda = 3/2$

Making the map ι_1 in section 9.2.1 explicit via canonical bases one gets ($\beta = -r + 5\chi - 1$)

$$\bigoplus_{w \in StB(\Sigma)} w H^0\Big(b, \mathcal{O}_b(r - \beta + [w]\,\chi - 2)\Big)^* \xrightarrow{\iota_1} \bigoplus_{w_2 \in StB(S^2\Sigma)} w_2\, H^0\Big(b, \mathcal{O}_b(-\beta + [w_2]\,\chi - 2)\Big)^* \quad \text{(G.1)}$$

(using (4.30)) or, with the notation $V := H^0(b, \mathcal{O}_b(1))$ and $S := Sym$ (cf. section I),

$$M_r : \qquad \widetilde{\Sigma}_w \odot S^{2r-5\chi-1}\, V^* \longrightarrow (\widetilde{S^2\,\Sigma})_{w_2 = w\, w'} \odot S^{r-5\chi-1}\, V^* \quad \text{(G.2)}$$

Here we use the symmetrized tensor product $\odot$ (cf. section I) and elements

$$w \in \{z, x, y\} = StB(\Sigma) \quad , \qquad w_2 \in \{z^2, zx, zy, x^2, xy, y^2\} = StB(S^2\Sigma) \quad \text{(G.3)}$$

of standard bases (StB) of $\Sigma = z\mathbf{C} \oplus x\mathbf{C} \oplus +y\mathbf{C}$ and $S^2\Sigma = Sym^2\Sigma$:

$$w \in H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s + [w]\,\chi F)\Big) \quad , \qquad w_2 \in H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(6s + [w_2]\,\chi F)\Big) \quad \text{(G.4)}$$

Furthermore[33] we made use of the notion of degree where $[w] := 2q + 3r$ for $w = z^p\, x^q\, y^r$

$$\widetilde{\Sigma}_w := \bigoplus_{w \in StB(\Sigma)} w\, S^{[w]\chi} V^* = z\,\mathbf{C} \oplus x\, S^{2\chi}\, V^* \oplus y\, S^{3\chi}\, V^* \quad \text{(G.5)}$$

$$(\widetilde{S^2\,\Sigma})_{w_2} := \bigoplus_{w_2 \in StB(S^2\Sigma)} w_2\, S^{[w_2]\chi} V^* \quad \text{(G.6)}$$

One finds in (G.2) that dim $lhs = 3(2r - 5\chi) + 5\chi = 6r - 10\chi = 6(r - 5\chi) + 20\chi =$ dim rhs. In this representation M_r (cf. section 9.2.1) is multiplication with an element

$$\begin{aligned} \tilde{\iota} \;=\; Cz + Bx + Ay \in H^0\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s + rF)\Big) &= \bigoplus_{w' \in StB(\Sigma)} w'\, H^0\Big(b, \mathcal{O}_b(r - [w']\,\chi)\Big) \quad \text{(G.7)} \\ &= \bigoplus_{w' \in StB(\Sigma)} w'\, S^{r - [w']\,\chi}\, V = \widetilde{\Sigma}_w \odot S^r\, V \end{aligned}$$

[33]By $\beta < 0 \Leftrightarrow r \geq 5\chi$ only the space belonging to z^2 can be zero-dimensional (for $r = 5\chi$; the other case $r = 0$ over $\mathbf{F_2}$ leads to a map between two zero-dimensional spaces where (4.16) is trivially fulfilled).

with the accompanying coefficients

$$C \in S^r\, V, \qquad B \in S^{r-2\chi}\, V, \qquad A \in S^{r-3\chi}\, V \tag{G.8}$$

For the mentioned bases one gets a block matrix (with first column *not* $(C, 0, B, A, 0, 0)^t$)

$$M_r \;: \qquad \begin{matrix} & \\ z^2 \\ zx \\ zy \\ x^2 \\ xy \\ y^2 \end{matrix} \begin{matrix} z \; x \; y \\ \begin{pmatrix} C & 0 & 0 \\ B & C & 0 \\ A & 0 & C \\ 0 & B & 0 \\ 0 & A & B \\ 0 & 0 & A \end{pmatrix} \end{matrix} \tag{G.9}$$

This is a square matrix of size $\big(6r-10\chi\big)\times\big(6r-10\chi\big)$. If, say, (cf. section I for the notation)

$$C = \sum_{i=0}^{r} c_i u^{r-i} v^i \in S^r\, V = Hom(S^r\, V^*, \mathbf{C}) \tag{G.10}$$

then we consider in (G.9) actually induced maps (or $(k+1)\times(k+r+1)$ - matrices)

$$C \underline{\odot}\, S^k\, V^* : S^{k+r}\, V^* \to S^k\, V^* \tag{G.11}$$

So, in (G.9), where we indicated the expansion coefficients in x, y, z accompanying the respective spaces in (G.2), each entry has to be suitable extended: a D in the line of w_2 stands actually for $D \underline{\odot}\, S^{r+\big([w_2]-5\big)\chi-1}\, V^*$ (where $D \in \{A, B, C\}$; cf. (I.10)). This is for $D \in S^{r-[w']\chi}\, V$ in the w - column a matrix of size $(r + \big([w_2]-5\big)\chi) \times (2r + \big([w]-5\big)\chi)$.

It will be shown that half of the determinants of all these matrices vanish, cf. section 7, the other ones (for $r \equiv \chi\,(2)$) have as factor the determinant of the following square matrix m_r of size $\frac{1}{4}\big(6r-10\chi\big) \times \frac{1}{4}\big(6r-10\chi\big)$ (for $r = 5\chi$ the first line is absent)

$$m_r \;: \qquad \begin{pmatrix} C \odot S^{\frac{r-5\chi}{2}-1}\, V^* \\ B \odot S^{\frac{r-5\chi}{2}-1+2\chi}\, V^* \\ A \odot S^{\frac{r-5\chi}{2}-1+3\chi}\, V^* \end{pmatrix} \qquad \left(\text{ where } \begin{matrix} C \in S^r\, V \\ B \in S^{r-2\chi}\, V \\ A \in S^{r-3\chi}\, V \end{matrix} \right) \tag{G.12}$$

m_r is mediated by multiplication with the element (G.7)). For $\chi = 1, r = 5$ one has

$$\begin{aligned} C &= c_5 u^5 + c_4 u^4 v + c_3 u^3 v^2 + c_2 u^2 v^3 + c_1 u v^4 + c_0 v^5 && \in S^r\, V = S^5\, V = Hom(S^5\, V^*, \mathbf{C}) \\ B &= b_3 u^3 + b_2 u^2 v + b_1 u v^2 + b_0 v^3 && \in S^{r-2\chi}\, V = S^3\, V = Hom(S^3\, V^*, \mathbf{C}) \\ A &= a_2 u^2 + a_1 u v + a_0 v^2 && \in S^{r-3\chi}\, V = S^2\, V = Hom(S^2\, V^*, \mathbf{C}) \end{aligned}$$

and gets for the map (G.12)

$$m_5 = \mathcal{D}_5 : \qquad \begin{pmatrix} b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \\ a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 \end{pmatrix} \tag{G.13}$$

For the case $\chi = 0, r = 2$ one gets for $C = C_2$ (and the same for $B = B_2$ and $A = A_2$)

$$C = c_2 u^2 + c_1 uv + c_0 v^2 \in S^2\, V = Hom(S^2\, V^*, \mathbf{C}) \tag{G.14}$$

and gets for the map (G.12)

$$m_3 \;= \mathcal{D}_3^{\chi=0} : \qquad \begin{pmatrix} c_2 & c_1 & c_0 \\ b_2 & b_1 & b_0 \\ a_2 & a_1 & a_0 \end{pmatrix} \tag{G.15}$$

H Bundles of $\lambda = 3/2$ over a $\chi = 0$ curve

Over $\mathbf{F_2}$ the curve b is movable along F in $\mathcal{E} = b \times F$ (i.e. the naive moduli space of motions, F, is of Euler number zero). Besides the issue of an integral over a moduli space the whole physical interpretation changes. As nevertheless some simplifications occur in this case we illustrate the general procedure (on a formal, i.e. purely mathematical level) also with examples from this case.

Note first that one has by (C.9) (with $-\beta - 2 \geq 0$ by $\beta = -(\lambda - \frac{1}{2})r - 1$ and (3.12))

$$\pi_{\mathcal{E}*}\mathcal{O}_{\mathcal{E}}(\alpha s + \beta F) = \bigoplus_{i=1}^{\alpha} \mathcal{O}_b(\beta) \tag{H.1}$$

As we will have cause to consider the higher cohomology groups in (4.11) we note also

$$\begin{aligned} H^1\Big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F\Big) &= H^1\Big(b,\, \bigoplus_{i=1}^{\alpha} \mathcal{O}_b(\beta)\Big) = \bigoplus_{i=1}^{\alpha} H^0\Big(b,\, \mathcal{O}_b(-\beta-2)\Big)^* \\ &= H^0\Big(F, \mathcal{O}_F(\alpha)\Big) \otimes H^0\Big(b,\, \mathcal{O}_b(-\beta-2)\Big)^* = \mathbf{C}^{\alpha}_{\mathbf{F}} \otimes Sym^{-\beta-2} V^* \end{aligned} \tag{H.2}$$

where $V := H^0\Big(b,\, \mathcal{O}_b(1)\Big) \cong \mathbf{C^2}$, generated by the first order monomials u and v (or linear polynomials in $t = u/v$, including a constant term). We use the identifications

$$H^0\Big(b, \mathcal{O}_b(p)\Big) = \mathbf{C}[t]_{\leq p} = Sym^p\, V = \mathbf{C}^{p+1} \tag{H.3}$$

$$H^0\Big(F, \mathcal{O}_F(q)\Big) = (\mathbf{C}[\mathbf{x}, \mathbf{y}]/rel)_{\leq q} = \mathbf{C^q_F} \tag{H.4}$$

($p \geq 0, q \geq 1$). The subscripts $|_{\leq p}$ and $|_{\leq q}$ indicate bounds on the degree (with deg $t = 1$ and deg $x = 2$, deg $y = 3$). We have divided out the relation $rel: \, y^2 = 4x^3 - g_2 x - g_3$ (we also apply the corresponding homogeneous version including z with deg $z = 0$).

With $\beta = -(\lambda - \frac{1}{2})r - 1 < 0$ here we have, according to (4.30), to consider the map

$$H^1\Big(\mathcal{E}, l(-F-c)\Big) \xrightarrow{\iota_1} H^1\Big(\mathcal{E}, l(-F)\Big) \tag{H.5}$$

or, explicitly with (H.2) and (4.30),

$$\begin{aligned} H^0\left(F, n\left(\lambda - \frac{1}{2}\right)\right) \otimes H^0\left(b, \left(\lambda + \frac{1}{2}\right) r - 1\right)^* \xrightarrow{\iota} H^0\left(F, n\left(\lambda + \frac{1}{2}\right)\right) \\ \otimes H^0\left(b, \left(\lambda - \frac{1}{2}\right) r - 1\right)^* \end{aligned}$$

which is a map between equidimensional spaces (by (4.15) where lhs $=0$ by $\beta<0$)

$$Sym^{(\lambda+\frac{1}{2})r-1}\,V^*\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}(\lambda-\frac{1}{2})} \xrightarrow{\ \iota\ } Sym^{(\lambda-\frac{1}{2})r-1}\,V^*\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}(\lambda+\frac{1}{2})} \tag{H.6}$$

making manifest the equal dimension $n(\lambda^2-\frac{1}{4})r$. The map ι is induced by multiplication with an element

$$\tilde{\iota}\in H^0\Big(\mathcal{E},\mathcal{O}_{\mathcal{E}}(c)\Big)\cong H^0(b,r)\otimes H^0(F,n)\cong Sym^r\,V\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}} \tag{H.7}$$

H.1 Example 3: the non-contributing case $r=1$

For $r=1$ we have $\lambda\in\frac{1}{2}+\mathbf{Z}^{(>\mathbf{0})}$ as $\lambda\in\mathbf{Z}$ needs n and r even. So take first $\lambda=3/2$. The identifications (H.2)

$$\begin{aligned}
H^1\Big(\mathcal{E},l(-F-c)\Big) &= H^1\Big(\mathcal{E},\mathcal{O}_{\mathcal{E}}(ns-(2r+1)F)\Big)\cong H^1\Big(b,\ \oplus_{i=1}^n\mathcal{O}_b(-2r-1)\Big)\\
&\cong \oplus_{i=1}^n H^0\Big(b,\ \mathcal{O}_b(2r-1)\Big)^*\\
H^1\Big(\mathcal{E},l(-F)\Big) &= H^1\Big(\mathcal{E},\mathcal{O}_{\mathcal{E}}(2ns-(r+1)F)\Big)\cong H^1\Big(b,\ \oplus_{i=1}^{2n}\mathcal{O}_b(-r-1)\Big)\\
&\cong \oplus_{i=1}^{2n} H^0\Big(b,\ \mathcal{O}_b(r-1)\Big)^*
\end{aligned}$$

show that the relevant map in (4.16) is given by the map between $2nr$-dimensional spaces

$$Sym^{2r-1}\,V^*\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}} \xrightarrow{\ \iota\ } Sym^{r-1}\,V^*\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{2n}} \tag{H.8}$$

with ι multiplication by $\tilde{\iota}\in Sym^r\,V\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}}$. For $r=1$ the map $\iota: V^*\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}}\rightarrow \mathbf{C}_{\mathbf{F}}^{\mathbf{2n}}$ has non-trivial kernel as multiplication with $\tilde{\iota}=\sum_{i=1}^n p_i\otimes x_i\in V\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}}$ maps[34] (for $y_j=x_j$)

$$V^*\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}}\ni \sum q_j\otimes y_j \longrightarrow \sum_{i,j} <p_i,p_j^{\perp}>(x_i\cdot x_j)=\sum_{i,j} p_i\wedge p_j\,(x_i\cdot x_j)=0 \tag{H.9}$$

for $q_j=p_j^{\perp}$ (reinterpreting V^* as V). So $W_b=0$ as (4.16) is violated; cf. section 6.3.1.

This reasoning explains algebraically the case $n=3$, found experimentally as example 3 in [8], also a special case of section 7; the argument extends to arbitrary n. In section 7 the same conclusion was argued even for any $\lambda=l+1/2$ with $l\in\mathbf{Z}^{>0}$ where $\beta=-lr-1$; one can consider the map (using the symmetrized tensor product $\odot$; cf. section I)

$$Sym^{(l+1)r-1}\,V^*\otimes \bigodot_1^{l}\mathbf{C}_{\mathbf{F}}^{\mathbf{n}} \longrightarrow Sym^{lr-1}\,V^*\otimes \bigodot_1^{l+1}\mathbf{C}_{\mathbf{F}}^{\mathbf{n}} \tag{H.10}$$

mediated by multiplication with $\tilde{\iota}=\sum p_i\otimes x_i\in Sym^r\,V\otimes \mathbf{C}_{\mathbf{F}}^{\mathbf{n}}$ and may contemplate a reasoning similar to above.

[34] Here the evaluation $<\,,\,>: V\times V^*\rightarrow \mathbf{C}$ is, via the canonical scalar product $<p,q>=p^{(1)}q^{(1)}+p^{(2)}q^{(2)}$, understood as a map $V\otimes V\longrightarrow \mathbf{C}$ (thus reinterpreting V^* as V, i.e. we take $p=p^{(1)}u+p^{(2)}v$ and $q=q^{(1)}u^*+q^{(2)}v^*$); furthermore, by combination with the map $V\ni q=(q^{(1)},q^{(2)})\longrightarrow q^{\perp}=(-q^{(2)},q^{(1)})\in V$, this can be understood as the map $V\wedge V\rightarrow\Lambda^2V\cong\mathbf{C}$ as one has $p\wedge q^{\perp}=p^{(1)}(q^{\perp})^{(2)}-p^{(2)}(q^{\perp})^{(1)}=p^{(1)}q^{(1)}+p^{(2)}q^{(2)}=<p,q>$.

Remark. There are some further obvious exceptional cases which do not contribute. For this note that for ι to be an isomorphism the map $H^0\Big(F, n(\lambda - \frac{1}{2})\Big) \otimes H^0(F, n) \longrightarrow H^0\Big(F, n(\lambda + \frac{1}{2})\Big)$ must be surjective; this excludes the cases $n(\lambda - \frac{1}{2}) = 1$, i.e. $n = 2, \lambda = 1$ as $H^0(F, 3)$ is not generated by $H^0(F, 1)$ and $H^0(F, 2)$, and $n = 2, \lambda = 3/2$ as $H^0(F, 4)$ is not generated by $H^0(F, 2)$ and $H^0(F, 2)$ (in both cases one can not get y from x).

I The symmetrized tensor product

Let us consider symmetrized tensor powers of the vector spaces

$$\Sigma = z\,\mathbf{C} \oplus x\,\mathbf{C} \oplus y\,\mathbf{C} \tag{I.1}$$

$$V = H^0\Big(b, \mathcal{O}(1)\Big) = u\,\mathbf{C} \oplus v\,\mathbf{C} \tag{I.2}$$

Symmetrized tensor powers of degree n mean that one considers expressions of order n (linear combinations of monomials which themselves consist of n factors of basis elements) within the polynomial algebra on the basis elements of the vector space. Thus one has for example (with $S^k := Sym^k$ and the dual basis elements u^*, v^* of V^*)

$$S^2\,\Sigma \;= z^2\,\mathbf{C} \oplus zx\,\mathbf{C} \oplus zy\,\mathbf{C} \oplus x^2\,\mathbf{C} \oplus xy\,\mathbf{C} \oplus y^2\,\mathbf{C} \tag{I.3}$$

$$S^3\,V^* = (u^*)^3\,\mathbf{C} \oplus (u^*)^2 v^*\,\mathbf{C} \oplus u^*(v^*)^2\,\mathbf{C} \oplus (v^*)^3\,\mathbf{C} \tag{I.4}$$

$$S^2\,V \;= u^2\,\mathbf{C} \oplus uv\,\mathbf{C} \oplus v^2\,\mathbf{C} \tag{I.5}$$

where we have used the polynomial notation for the composed elements. In general we denote by $\odot$ the symmetric tensor product. So one has (with $\cdot$ for evaluation; let $n \geq m$)

$$S^a\,V \odot S^b\,V = S^{a+b}\,V \tag{I.6}$$

$$S^n\,V^* \cdot S^m\,V = S^{n-m}\,V^* \tag{I.7}$$

For example one has

$$u^{*3}v^{*2} \cdot (au^2 + buv + cv^2) = (cu^{*2} + bu^*v^* + av^{*2})u^* \tag{I.8}$$

We will apply the prescription $\cdot \longrightarrow \cdot \odot\; S^k\,V^*$ (i.e. symmetric product with $S^k\,V^*$), not only to spaces, but also, functorially, to maps $f : A \longrightarrow B$. Then we use the notation

$$f \,\underline{\odot}\, S^k\,V^* : \quad A \odot S^k\,V^* \longrightarrow B \odot S^k\,V^* \tag{I.9}$$

If, to give an example, $C = c_0u^3 + c_1u^2v + c_2uv^2 + c_3v^3 \in S^3\,V = Hom(S^3\,V^*, \mathbf{C})$, then one gets for $C \,\underline{\odot}\, S^2\,V^* : \; S^5\,V^* \longrightarrow S^2\,V^*$ the matrix representation

$$C \underline{\odot} S^2\,V^* = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & 0 & 0 \\ 0 & c_0 & c_1 & c_2 & c_3 & 0 \\ 0 & 0 & c_0 & c_1 & c_2 & c_3 \end{pmatrix} \tag{I.10}$$

I.1 Interpretation of the resultant criterion

Let $f = \sum_{i=0}^{m} a_i z^i$ and $g = \sum_{i=0}^{n} b_i z^i$ polynomials in the complex variable z (with $a_m \neq 0, b_n \neq 0$; homogeneous polynomials are treated analogously). Euler and Sylvester remarked that f and g have a common zero z_* precisely if a certain determinant vanishes, the resultant $Res(f,g)$ of f and g, a polynomial of degree $m+n$ in the coefficients a_i and b_i. If we take for example $m=5$ and $n=2$ the determinant in question is that of

$$\begin{pmatrix} a_5 & 0 & b_2 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & b_1 & b_2 & 0 & 0 & 0 \\ a_3 & a_4 & b_0 & b_1 & b_2 & 0 & 0 \\ a_2 & a_3 & 0 & b_0 & b_1 & b_2 & 0 \\ a_1 & a_2 & 0 & 0 & b_0 & b_1 & b_2 \\ a_0 & a_1 & 0 & 0 & 0 & b_0 & b_1 \\ 0 & a_0 & 0 & 0 & 0 & 0 & b_0 \end{pmatrix} \tag{I.11}$$

(or of its transpose). For the assertion $f = (z-z_*)\tilde{f}$, $g = (z-z_*)\tilde{g}$ is equivalent to have

$$f\tilde{g} = g\tilde{f} \tag{I.12}$$

for some polynomials $\tilde{f}, \tilde{g}$ of degree $m-1$ and $n-1$ ($\tilde{f}, \tilde{g} \neq 0$). For the equivalence note that clearly not all linear factors of f can come from $\tilde{f}$; the other direction is obvious.

From the equality (I.12) of polynomials of degree $m+n-1$ the classical reasoning proceeds by comparison of their $m+n$ coefficents to the indicated matrix of a system of $m+n$ linear equations. More in the spirit of our investigation is to argue as follows. For $f \in S^m V \cong Hom(\mathbf{C}, S^m V)$ one has the map given by multiplication with f

$$f \underline{\odot} S^k V \in Hom(S^k V, S^{k+m} V) \tag{I.13}$$

(correspondingly for g). With this definition consider the following map

$$\underline{m} : S^{n-1}V \oplus S^{m-1}V \ni (p,q) \longrightarrow (f \underline{\odot} S^{n-1} V)\, p + (g \underline{\odot} S^{m-1} V)\, q = fp + gq \in S^{m+n-1}V$$

Now (I.12) just expresses the fact $\underline{m}(\,(\tilde{g}, -\tilde{f})\,) = 0$, i.e. that the map $\underline{m}$ is not injective. However $\underline{m}$ has just (I.11) as matrix and $Res(f,g) = \det \underline{m}$.

Example 1 has $Res(B_2, A_1) = \det(B \oplus A \underline{\odot} V^*)$ with $B \oplus A \underline{\odot} V^* \in Hom(S^2 V^*, \mathbf{C} \oplus V^*)$, cf. (A.5), and $Res(C_4, A_1) = \det(C \oplus A \underline{\odot} S^3 V^*)$, cf. (A.11), (A.12).

I.2 Polynomials having more that one root in common

The classical case gives a determinantal criterion for two polynomials to have one root in common. We will also be interested in the case of having more roots in common.

Let us assume that (where $\deg \tilde{f} = m-2, \deg \tilde{g} = n-2$)

$$f = (z-z_1)(z-z_2)\tilde{f} \tag{I.14}$$

$$g = (z-z_1)(z-z_2)\tilde{g} \tag{I.15}$$

As before an precise criterion for this case is a relation

$$fp = gq \tag{I.16}$$

where $\deg \tilde{q} = m-2, \deg \tilde{p} = n-2$. Equivalently consider the map

$$\underline{m}: \; S^{n-2}V \oplus S^{m-2}V \ni (p,q) \longrightarrow fp + gq \in S^{m+n-2}V \tag{I.17}$$

By (I.16) we search a criterion for $\ker \underline{m} \neq 0$. As $\dim \text{source}\, \underline{m} = m+n-2$ and $\dim \text{target}\, \underline{m} = m+n-1$ we should consider $\underline{m}$ restricted (in its target) to its image.

To describe this in greater detail we focus on the example $m=2, n=3$ which is of importance in section 9.2.1 (cf. example 2). So let $f = \sum_{i=0}^{2} f_i z^i, g = \sum_{i=0}^{3} g_i z^i$ and

$$\underline{m}: \; V \oplus \mathbf{C} \ni (p,q) \longrightarrow fp + gq \in S^3V \tag{I.18}$$

(where $p = p_1 z + p_0, q = q_0$) of matrix $Mat[g,f]^{tr}$ (cf. (I.20) without first row). Now

$$h = \sum_{i=0}^{3} h_i z^i \in im\; \underline{m} \Longleftrightarrow \det Mat[h,g,f] = -\sum_{i=0}^{3} (-1)^i h_i M_i = 0 \tag{I.19}$$

where the matrix of the map $h \oplus g \oplus f \underline{\circ} V^* : S^3V^* \longrightarrow \mathbf{C} \oplus \mathbf{C} \oplus V^*$ occurs

$$Mat[h,g,f] = \begin{pmatrix} h_3 & h_2 & h_1 & h_0 \\ g_3 & g_2 & g_1 & g_0 \\ f_2 & f_1 & f_0 & 0 \\ 0 & f_2 & f_1 & f_0 \end{pmatrix} \tag{I.20}$$

with minors M_i associated to the development w.r.t. the first row. To see (I.19) one may eliminate p_1, p_0, q_0 from the coefficents of a typical image element to get directly $\sum_{i=0}^{3} (-1)^i h_i M_i = 0$. Note that one gets from taking $h = f$ or fz the relations $f_2 M_2 = -f_0 M_0 + f_1 M_1$ and $f_2 M_3 = -f_0 M_1 + f_1 M_2$ between the minors. A basis for the three-dimensional image of $\underline{m}$ is given by fz, f and g (which explains (I.19) in an elementary way). For these elements to become linearly dependent one finds by direct inspection

$$ker\; \underline{m} \neq 0 \Longleftrightarrow M_0 = 0 = M_1 \tag{I.21}$$

A common root of M_0 and M_1 implies of course a second order zero of the resultant as

$$Res(g,f) \;=\; -\frac{1}{f_1} \det \begin{pmatrix} M_3 & M_2 \\ M_1 & M_0 \end{pmatrix} \;=\; \frac{1}{f_2^2} \det \begin{pmatrix} f_0 & f_1 & f_2 \\ M_1 & M_0 & 0 \\ 0 & M_1 & M_0 \end{pmatrix} \tag{I.22}$$

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